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ON THE CONVERGENCE OF SOME ITERATION PROCESSES IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. For the approximation of fixed points of a nonexpansive operator T in a uniformly convex Banach space E the convergence of the Mann-Toeplitz iteration $x_{n+1} = \alpha_n T(x_n) + (1 - \alpha_n)x_n$ is studied. Strong convergence is established for a special class of operators T. Via regularization this result can be used for general nonexpansive operators, if E possesses a weakly sequentially continuous duality mapping. Furthermore strongly convergent combined regularization-iteration methods are presented.

Throughout this note, let $(E, |\cdot|)$ be a uniformly convex Banach space, let C be a nonempty closed convex subset of E. Let T: $C \to C$ denote a nonexpansive operator, i.e. $|T(x) - T(y)| \leqslant |x - y|$ holds for all $x, y \in C$. To approximate a fixed point of T we define the following iterative method (Mann-Toeplitz process) by

$$x_1 \in C, \quad x_{n+1} = \alpha_n T(x_n) + (1 - \alpha_n)x_n, \qquad \alpha_n \in [0, 1] \; (n \geqslant 1). \quad (1)$$

We make the assumptions that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\alpha_n \in [0, b]$ with $b \in (0, 1)$ for almost all positive integers n.

Let us recall that any mapping J: $E \to E^*$ which fulfills

$$(J(u), u) = |J(u)| \cdot |u|, \qquad |J(u)| = |u|$$

for all $u \in E$ is termed a duality mapping. We verify easily (see also [3, Theorem 8.9]) that the nonexpansive operator T satisfies

$$(x - y - T(x) + T(y), J(x - y)) \geqslant 0$$

for all $x, y \in C$. This means that the operator $S := I - T$ is accretive. Now we call an operator S: $C \to E$ φ-accretive if there exists a function φ: $[0, \infty) \to [0, \infty)$ strictly increasing with $\varphi(0) = 0$, and $\lim_{t\to\infty}\varphi(t) = \infty$ such that (cf. [1] for monotone operators)

$$(S(x) - S(y), J(x - y)) \geqslant [\varphi(|x|) - \varphi(|y|)] \cdot [|x| - |y|] \qquad \forall x, y \in C. \quad (2)$$

If S satisfies the stronger condition

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$$(S(x) - S(y), J(x-y)) \geqslant \varphi(|x-y|) \cdot |x-y| \quad \forall x, y \in C, \tag{3}$$

then S is called uniformly φ-accretive.

THEOREM 1. *Let the fixed point set F of the operator T be nonempty. Suppose the operator $S = I - T$ is φ-accretive. Then the Mann-Toeplitz sequence $\{x_n\}$ converges strongly to the unique fixed point $p \in F$.*

PROOF. Since for any fixed $p \in F$

$$|x_{n+1} - p| \leqslant |x_n - p|,$$

the sequence $\{x_n\}$ is bounded. Now assume p_1, p_2 belong to F. By (2) it follows that

$$[\varphi(|p_1|) - \varphi(|p_2|)] \cdot [|p_1| - |p_2|] = 0,$$

therefore $|p_1| = |p_2|$. On the other hand F is convex, and is contained in the uniformly convex space E. So F reduces to a single point p.

According to a result of Ishikawa [7, Lemma 2] $S(x_n)$ converges strongly to zero, and by a theorem due to Browder [3, Theorem 8.4, p. 103] the sequence $\{x_n\}$ converges weakly to the unique fixed point p. Since J is a bounded operator, the sequence $\{J(x_n - p)\}$ remains bounded. Hence (2) implies that

$$[\varphi(|x_n|) - \varphi(|p|)] \cdot [|x_n| - |p|] \to 0 \qquad (n \to \infty).$$

It follows easily (cf. [1, p. 61]) that $|x_n| \to |p|$. This yields the claimed norm convergence of the sequence $\{x_n\}$ in the uniformly convex space E.

Since strictly contractive operators are uniformly φ-accretive, Theorem 3 contains a result in [5, Theorem 1]. Let us note that in a Hilbert space E the gradient f' of a Gateaux differentiable, convex functional f is φ-accretive, if

$$f((x+y)/2) \leqslant \tfrac{1}{2} f(x) + \tfrac{1}{2} f(y) - [\varphi(|x|) - \varphi(|y|)] \cdot [|x| - |y|]$$

is valid for all $x, y \in C$. This fact follows from the estimate

$$(f'(x) - f'(y), x - y) \geqslant 2[f(x) + f(y) - 2f((x+y)/2)].$$

Furthermore Theorem 1 remains true, if the inequality (2) is only assumed to hold for all $x \in C$ and all $p \in F$, i.e. if

$$(x - T(x), J(x-p)) \geqslant [\varphi(|x|) - \varphi(|p|)] \cdot [|x| - |p|] \quad \forall x \in C, \forall p \in F$$

is assumed. Operators T: $C \to C$ that satisfy for any $x, y \in C$

$$\begin{aligned} |T(x) - T(y)| \leqslant\ & a_1|x - T(x)| + a_2|y - T(y)| \\ & + a_3|x-y| + a_4|x - T(y)| + a_5|y - T(x)| \end{aligned}$$

with $a_i \geqslant 0$ $(i = 1, \ldots, 5)$ and $\sum_{i=1}^{5} a_i \leqslant 1$ belong to this class, provided

$$2a_1 + a_3 + a_4 + a_5 < 1$$

holds.

Even Theorem 1 can be applied to general nonexpansive operators T, for then the operators $S_\varepsilon = (1-\varepsilon)(I - T) + \varepsilon R$ $(\varepsilon > 0)$ inherit the (uniform) φ-accretiveness from the "regularization operator" R. This observation motivates the following study of the regularization method involved.

THEOREM 2. *Let the fixed point set F of T in C be nonempty. Let $R\colon C \to E$ be a continuous, bounded operator. Suppose R is uniformly φ-accretive with respect to an odd, weakly sequentially continuous duality mapping $J\colon E \to E^*$. Choose positive reals δ_k, and $\varepsilon_k \in (0, 1)$ with $\lim_{k\to\infty}\varepsilon_k = 0$, and $\lim_{k\to\infty} \delta_k\varepsilon_k^{-1} = 0$. If the approximate solutions $\tilde{y}_k \in C$ satisfy*

$$|(1 - \varepsilon_k)(I - T)(\tilde{y}_k) + \varepsilon_k R(\tilde{y}_k)| \leqslant \delta_k,$$

then the sequence $\{\tilde{y}_k\}$ converges strongly to a fixed point $\hat{p}$, which is uniquely determined by the variational inequality

$$(R(\hat{p}), J(\hat{p} - p)) \leqslant 0 \quad \forall p \in F. \tag{4}$$

PROOF. Let $p \in F$, and set

$$\beta_k = ((1 - \varepsilon_k)(I - T)(\tilde{y}_k) + \varepsilon_k R(\tilde{y}_k), J(\tilde{y}_k - p)).$$

We notice that $|\beta_k| \leqslant |\tilde{y}_k - p|\delta_k$. Since $I - T$ is accretive, it follows

$$(R(\tilde{y}_k), J(\tilde{y}_k - p)) \leqslant \beta_k\varepsilon_k^{-1}. \tag{5}$$

Let us prove the boundedness of the sequence $\{y_k\}$. On account of (5), (3) we conclude

$$\begin{aligned}\beta_k\varepsilon_k^{-1} + |R(p)| \cdot |\tilde{y}_k - p| &\geqslant \beta_k\varepsilon_k^{-1} - (R(p), J(\tilde{y}_k - p))\\ &\geqslant (R(\tilde{y}_k) - R(p), J(\tilde{y}_k - p))\\ &\geqslant \varphi(|\tilde{y}_k - p|) \cdot |\tilde{y}_k - p|.\end{aligned}$$

We may assume without loss of generality that $|\tilde{y}_k - p|$ is positive, and hence we obtain

$$\delta_k\varepsilon_k^{-1} + |R(p)| \geqslant \varphi(|\tilde{y}_k - p|).$$

The boundedness of $\{\tilde{y}_k\}$ is immediate, and with a constant c_p, dependent only on $p \in F$, (5) reads

$$(R(\tilde{y}_k), J(\tilde{y}_k - p)) \leqslant c_p\delta_k\varepsilon_k^{-1}. \tag{6}$$

As R is a bounded operator, $\varepsilon_k R(\tilde{y}_k)$ converges to zero. Since the nonexpansive operator T is also bounded, and $\delta_k \to 0$, we conclude that $(I - T)(\tilde{y}_k)$ converges strongly to zero. By a theorem due to Browder **[3**, Theorem 8.4] all weak limit points of $\{\tilde{y}_k\}$, which exist by the boundedness of $\{\tilde{y}_k\}$, belong to F. Let $\tilde{y} = \text{w-lim}_{i\to\infty} \tilde{y}_{k_i}$; then (3) and (6) imply that

$$c_{\tilde{y}}\delta_{k_i}\varepsilon_{k_i}^{-1} - (R(\tilde{y}), J(\tilde{y}_{k_i} - \tilde{y})) \geqslant \varphi(|\tilde{y}_{k_i} - \tilde{y}|) \cdot |\tilde{y}_{k_i} - \tilde{y}|.$$

Since J is weakly sequentially continuous, we see at once that $\tilde{y} = \lim_{i\to\infty} y_{k_i}$, and (6) results in the claimed inequality (4). Finally we have to show that this inequality uniquely determines $\hat{p} \in F$, thus proving the convergence of the entire sequence $\{\tilde{y}_k\}$. Fix some $p_1, p_2 \in F$ that satisfy (4), then

$$(R(p_1), J(p_1 - p_2)) \leqslant 0, \qquad -(R(p_2), J(p_1 - p_2)) \leqslant 0.$$

The summation of both these inequalities yields $p_1 = p_2$, since R is uniformly φ-accretive.

If the regularization operator R is only φ-accretive, then similar but more involved arguments show that the sequence $\{\tilde{y}_k\}$ is bounded, every weak limit point of $\{\tilde{y}_k\}$ is also a strong limit point, and every limit point belongs to the fixed point set and satisfies (4). But as J is not linear, unless E is a Hilbert space, the set of points which fulfill (4) is generally not convex; therefore uniqueness cannot be obtained as in the proof of Theorem 1.

The approximate solutions $\tilde{y}_k$ can be constructed by finitely many Mann-Toeplitz iterations for the operator $T_k = (1 - \varepsilon_k)T + \varepsilon_k(I - R)$ by Theorem 1, provided $I - R$: $C \to C$ is nonexpansive. If furthermore C is bounded, fixed points of T and of each T_k exist.

The simplest regularization method is given by $R(x) = x - x^0$, x^0 fixed in C. In this case Reich [**10**, Corollary] has already established the strong convergence of the exact solutions $y_k = (1 - \varepsilon_k)T(y_k) + \varepsilon_k x^0$ $(\delta_k = 0)$ to a fixed point of T under similar conditions. In view of the inequality (4) which is achieved by regularization, other choices of R should be taken into consideration.

Inspired by the work of Bruck [**4**], and Halpern [**6**] we combine in conclusion Mann-Toeplitz iteration and regularization to the following iteration process

$$z_1 \in C, \; z_{m+1} = \alpha_m(1 - \varepsilon_m)T(z_m) + \alpha_m \varepsilon_m U(z_m) + (1 - \alpha_m)z_m. \tag{7}$$

Here we require that U: $C \to C$ is a strict contraction with contraction constant $q \in [0, 1)$. Clearly $R = I - U$ is then uniformly φ-accretive with $\varphi(t) = (1 - q)t$. Let us note that the choice $U(z) = \mathring{z}$, $\mathring{z}$ fixed in C, reduces (7) with $\varepsilon_m = \Theta_m(1 + \Theta_m)^{-1}$, $\alpha_m = \lambda_m(1 + \Theta_m)$ to the iteration method which is considered in [**4**, p. 123], and is also contained in the projection-iteration method of Bakusinskii and Poljak [**2**, Theorem 3D] for the solution of variational inequalities in Hilbert spaces.

The subsequent results hold in arbitrary Banach spaces E.

THEOREM 3. *Let C be a bounded closed convex subset of E. Suppose the sequence $\{y_i\}$ converges to a fixed point p of T, where y_i is given by*

$$y_i = (1 - \varepsilon_i)T(y_i) + \varepsilon_i U(y_i), \tag{8}$$

with $\varepsilon_i \in (0, 1]$, $\{\varepsilon_i\}$ monotonically decreasing to zero. If the two sequences $\{\varepsilon_m\}$ and $\{\alpha_m\}$, contained in $(0, 1]$, satisfy with some strictly increasing sequence $\{m(k)\}$ of positive integers

$$\liminf_{k\to\infty} \varepsilon_{m(k)} \sum_{j=m(k)}^{m(k+1)} \alpha_j > 0, \tag{9}$$

$$\lim_{k\to\infty} \left[\varepsilon_{m(k)} - \varepsilon_{m(k+1)}\right] \cdot \sum_{j=m(k)}^{m(k+1)} \alpha_j = 0, \tag{10}$$

then the sequence $\{z_m\}$ generated by (7) converges to p.

PROOF. We follow the pattern of proof in [**4**, pp. 117–119], but we dispense with inner product structure.

Banach's fixed point theorem guarantees existence and uniqueness of each y_i. We calculate for $m > i \geqslant 1$

$$\begin{aligned} |z_m - y_i| &= |\alpha_{m-1}(1-\varepsilon_{m-1})T(z_{m-1}) + \alpha_{m-1}\varepsilon_{m-1}U(z_{m-1}) \\ &\qquad + (1-\alpha_{m-1})z_{m-1} - y_i| \\ &\leqslant (1-\alpha_{m-1})|z_{m-1} - y_i| \\ &\quad + \alpha_{m-1}|(1-\varepsilon_{m-1})T(z_{m-1}) + \varepsilon_{m-1}U(z_{m-1}) \\ &\qquad - (1-\varepsilon_i)T(y_i) - \varepsilon_i U(y_i)| \\ &\leqslant [1 - \alpha_{m-1} + \alpha_{m-1}(1-\varepsilon_i) + \alpha_{m-1}\varepsilon_i q] \cdot |z_{m-1} - y_i| \\ &\quad + \alpha_{m-1}(\varepsilon_i - \varepsilon_{m-1}) \cdot |T(z_{m-1}) - U(z_{m-1})|. \end{aligned}$$

Hence

$$|z_m - y_i| \leqslant [1 - \alpha_{m-1}\varepsilon_i(1-q)] \cdot |z_{m-1} - y_i| + \alpha_{m-1}(\varepsilon_i - \varepsilon_{m-1})c, \quad (11)$$

with some constant c, because T and U are self-mappings of the bounded set C. Since the exp function is convex and therefore $\exp(t) - 1 \geqslant t$ holds, it follows that

$$|z_m - y_i| \leqslant \exp[-\alpha_{m-1}\varepsilon_i(1-q)] \cdot |z_{m-1} - y_i| + c\alpha_{m-1}(\varepsilon_i - \varepsilon_{m-1}).$$

By induction we conclude

$$|z_m - y_i| \leqslant \exp\left[-\varepsilon_i(1-q)\sum_{j=i}^{m-1}\alpha_j\right] \cdot |z_i - y_i| + c\sum_{j=i}^{m-1}\alpha_j(\varepsilon_i - \varepsilon_j).$$

On account of $\varepsilon_i - \varepsilon_j \leqslant \varepsilon_i - \varepsilon_m$ for $j \leqslant m$ we weaken this estimate to

$$|z_m - y_i| \leqslant \exp\left[-(1-q)\varepsilon_i\sum_{j=i}^{m-1}\alpha_j\right] \cdot |z_i - y_i| + c(\varepsilon_i - \varepsilon_m)\sum_{j=i}^{m}\alpha_j.$$

Starting with this inequality, which corresponds to inequality (12) in [4], one can easily adapt the arguments in [**4**, pp. 118–119] to conclude the proof. The details are omitted.

Examples of sequences $\{\alpha_n\}$ and $\{\varepsilon_n\}$ that satisfy both the conditions (9) and (10) are given by $\alpha_n = 1/n$, and $\varepsilon_n = 1/\log\log n$ for $n > 2$, or by $\alpha_n = n^{-p}$ and $\varepsilon_n = n^{-q}$ for $n > 2$, provided $0 < p < 1$ and $0 < q < 1 - p$ holds (see Bruck [**4**, p. 125]). Also one can choose $\alpha_n = \lambda \in (0, 1]$ fixed, and $\varepsilon_{n(k)} = \varepsilon_{n(k)+1} = \cdots = \varepsilon_{n(k+1)-1} = k^{1-p}$, where $n(k) \sim k^p$ and $p > 1$. The resulting iteration process is then related to [**6**, Theorem 4]. Furthermore the choice $\alpha_n = \lambda$, $\varepsilon_n = n^{-p}(1 + n^{-p})^{-1}$, $p \in (0, 1)$ with $n(k) \sim k^r$, $r = (1 - p)^{-1}$ leads to the example D of Theorem 3D with $P_K = I$ in [**2**].

Simpler sufficiency criteria for strong convergence are provided by

THEOREM 4. *Let C be a bounded closed convex subset of E. Suppose, the sequence $\{y_i\}$, given by* (8), *converges to a fixed point p of T, where $\varepsilon_i \in (0, 1]$ and* $\lim_{i\to\infty} \varepsilon_i = 0$. *If the two sequences $\{\varepsilon_i\}$ and $\{\alpha_i\}$, contained in* (0, 1], *satisfy*

$$\sum_i \alpha_i\varepsilon_i = +\infty, \quad (12)$$

$$|\varepsilon_{i-1} - \varepsilon_i| \cdot \varepsilon_i^{-1} = o(\alpha_i \varepsilon_i), \tag{13}$$

then the sequence $\{z_m\}$ *generated by* (7) *converges to* p.

PROOF. We simplify (11) to

$$\delta_{m+1} := |z_{m+1} - y_m| \leqslant [1 - \alpha_m \varepsilon_m (1-q)] |z_m - y_m|.$$

On the other hand we have

$$|y_i - y_{i-1}| \leqslant (1-\varepsilon_i)|y_i - y_{i-1}| + |\varepsilon_{i-1} - \varepsilon_i| \cdot |T(y_{i-1})| + \varepsilon_i q |y_i - y_{i-1}| + |\varepsilon_{i-1} - \varepsilon_i| \cdot |U(y_{i-1})|,$$

hence with some constant d

$$|y_i - y_{i-1}| \leqslant d \cdot |\varepsilon_{i-1} - \varepsilon_i| \varepsilon_i^{-1}.$$

Thus we obtain with $\chi_m = (1-q)\alpha_m \varepsilon_m$, $\gamma_m = d \cdot |\varepsilon_{m-1} - \varepsilon_m| \varepsilon_m^{-1} \cdot \chi_m^{-1}$

$$\delta_{m+1} \leqslant (1 - \chi_m)\delta_m + \chi_m \gamma_m,$$

and consequently for arbitrary $j \geqslant 0$

$$\delta_{m+j+1} \leqslant \left(\prod_{i=m}^{m+j} (1 - \chi_i) \right) \delta_m + \sum_{i=m}^{m+j} \left(\prod_{k=i+1}^{m+j} (1 - \chi_k) \right) \chi_i \gamma_i. \tag{14}$$

By (12), $\Pi(1 - \chi_i)$ diverges to zero. Since

$$\sum_{i=m}^{m+j} \left(\prod_{k=i+1}^{m+j} (1 - \chi_k) \right) \chi_i \leqslant 1$$

for any j and $\lim_{i \to \infty} \gamma_i = 0$ by (13), a well-known theorem of Toeplitz (cf. [**8**, p. 75]) implies that the second term in (14) converges to zero ($j \to \infty$), too. Thus we arrive at $\lim_{i \to \infty} \delta_i = 0$.

This result is closely related to Theorem 3D in [**2**] and contains (choose $U(z) = y$ fixed, $\alpha_i = 1$ fixed) a recent result of Lions [**9**, Theorem **1**].

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