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A GENERAL RESULT REGARDING THE GROWTH OF SOLUTIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. In this paper, we treat first-order algebraic differential equations whose coefficients are arbitrary complex-valued functions on an interval $[x_0, +\infty)$, and we obtain an estimate on the growth of all real-valued solutions on $[x_0, +\infty)$. Our result includes, as a very special case, the well-known result of Lindelöf for polynomial coefficients.

1. Introduction. In this paper we treat first-order algebraic differential equations, $\Omega(x, y, y') = 0$, where $\Omega(x, y, y') = \sum f_{ij}(x) y^i (y')^j$ is a polynomial in y and y' whose coefficients $f_{ij}(x)$ are arbitrary complex-valued functions defined on an interval $[x_0, +\infty)$. Our main result (§2 below) provides a growth estimate for any real-valued solution of $\Omega(x, y, y') = 0$ on $[x_0, +\infty)$, in terms of a monotone nondecreasing majorant $M(x)$ of all the functions $|f_{ij}(x)|$, and a positive monotone nonincreasing function $N(x)$ which is majorized by $|f_{p-m,m}(x)|$, where p is the total degree of Ω in y and y', and where m is the maximum integer j for which the coefficient $f_{p-j,j}(x)$ is not identically zero. (Thus our result applies only to those equations $\Omega(x, y, y') = 0$, for which the "leading" coefficient $f_{p-m,m}(x)$ is nowhere zero on some interval $[x_0, +\infty)$.) The estimate is actually obtained in terms of the function $U(x) = (m + 1)M(x)/N(x)$, and our result states that if $y(x)$ is a real-valued solution of $\Omega(x, y, y') = 0$ on an interval $[x_0, +\infty)$, then,

$$y(x) = O\left(\exp\left(U(x) + \int_{x_0}^{x} U(t)\,dt\right)\right) \quad \text{as } x \to +\infty. \tag{1}$$

(Actually, we obtain a slight improvement of (1) in the case where $|y(x)|$ grows sufficiently rapidly.) It should be pointed out that in the case where $|y(x)|$ does not grow sufficiently rapidly, we require that $U(x)$ be differentiable and have the property that for any $\varepsilon > 0$, the function $U'(x)/\exp(\varepsilon U(x))$ tends to zero as $x \to +\infty$. Of course, this property is possessed by the functions which usually serve as majorants namely $\exp_k(x^A)$, where $\exp_k$ is the kth iterate of the exponential function, and A is a constant. (See also §6 below.)

As a very special case of our result, we obviously obtain another proof of

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Lindelöf's theorem [**6**] which states that if all the coefficients $f_{ij}(x)$ are polynomials, then any real-valued solution $y(x)$ on an interval $[x_0, +\infty)$ satisfies an inequality of the form, $|y(x)| \leqslant \exp(x^A)$ for all sufficiently large x, where A is a constant. However, we emphasize that our result will provide a growth estimate for the real-valued solutions of *any* first-order algebraic differential equation for which we possess the upper and lower estimates $M(x)$ and $N(x)$.

We conclude with three remarks. First, no analogue of our result can hold for either complex-valued solutions of first-order algebraic differential equations or real-valued solutions of higher order algebraic differential equations, since such solutions of such equations (even with constant coefficients) can dominate any preassigned function on $(0, +\infty)$ at a sequence of x tending to $+\infty$. (See Vijayaraghavan [**9**] and Vijayaraghavan, Basu and Bose [**10**]. In addition, we mention that the construction in [**10**] was extended to show [**1**, §14, p. 53] that increasing solutions of third-order algebraic differential equations with constant coefficients can also have arbitrarily rapid growth.) Secondly, for a survey of the classical results on the growth of real-valued solutions of algebraic differential equations, we refer the reader to Chapter 5 of R. Bellman's book [**2**]. Finally, for the reader who is interested in the question of growth of real-valued solutions of algebraic difference equations, algebraic functional equations, and algebraic differential-difference equations, we refer the reader to the papers of K. Cooke [**3**], [**4**], O. Lancaster [**5**], and S. M. Shah [**7**], [**8**].

2. We now state our main result.

THEOREM. *Let $\Omega(x, y, y') = \sum f_{ij}(x) y^i (y')^j$ be a polynomial in y and y' whose coefficients $f_{ij}(x)$ are complex-valued functions defined on an interval $[x_0, +\infty)$, and let some coefficient be not identically zero. Let $p = \max\{i + j: f_{ij} \not\equiv 0\}$. Let $M(x)$ be a monotone nondecreasing function on $[x_0, +\infty)$ such that for each (i, j),*

$$|f_{ij}(x)| \leqslant M(x) \quad \textit{for all } x \geqslant x_0. \tag{2}$$

Let $m = \max\{j: f_{p-j,j} \not\equiv 0\}$, and assume $f_{p-m,m}(x)$ is nowhere zero on $[x_0, +\infty)$. Assume further that there exists a monotone nonincreasing function $N(x)$ on $[x_0, +\infty)$ such that

$$|f_{p-m,m}(x)| \geqslant N(x) > 0 \quad \textit{for all } x \geqslant x_0. \tag{3}$$

Set $U(x) = (m+1)M(x)/N(x)$. Let $y(x)$ be a real-valued function on $[x_0, +\infty)$ which has a continuous first derivative and which satisfies $\Omega(x, y(x), y'(x)) \equiv 0$ on $[x_0, +\infty)$. Then:

(a) *If for each δ in $(0, 1)$, the function $N(x)|y(x)|^\delta / M(x)$ tends to $+\infty$ as $x \to +\infty$, then*

$$y(x) = O\left(\exp\left(\int_{x_0}^{x} U(t)\, dt\right)\right) \quad \textit{as } x \to +\infty. \tag{4}$$

(b) *Suppose that for some* δ *in* (0, 1), *the function* $N(x)|y(x)|^{\delta}/M(x)$ *does not tend to* $+\infty$ *as* $x \to +\infty$. *In addition, assume that* $U(x)$ *is differentiable and satisfies the condition that for any* $\varepsilon > 0$, *the function* $U'(x)/\exp(\varepsilon U(x))$ *tends to zero as* $x \to +\infty$. *Then*,

$$y(x) = O\left(\exp\left(U(x) + \int_{x_0}^{x} U(t)\,dt\right)\right) \quad \text{as } x \to +\infty. \tag{5}$$

3. We first establish some notation, and then we will prove parts (a) and (b) separately. By dividing the relation $\Omega(x, y(x), y'(x)) = 0$ through by $(y(x))^p$ (at any point where $y(x) \neq 0$), we obtain,

$$\Lambda(x) = \Phi(x), \tag{6}$$

where

$$\Lambda(x) = \sum_{j=0}^{m} f_{p-j,j}(x)(y'(x)/y(x))^{j}, \tag{7}$$

$$\Phi(x) = -\sum_{i+j<p} h_{ij}(x), \tag{8}$$

and where,

$$h_{ij}(x) = f_{ij}(x)(y'(x)/y(x))^{j}(y(x))^{i+j-p}, \quad \text{for } i + j < p. \tag{9}$$

From (7), we can write,

$$\Lambda(x) = f_{p-m,m}(x)(y'(x)/y(x))^{m}\left(1 + \sum_{j=0}^{m-1} \Psi_j(x)\right), \tag{10}$$

where for $j = 0, 1, \ldots, m-1$,

$$\Psi_j(x) = (f_{p-j,j}(x)/f_{p-m,m}(x))(y'(x)/y(x))^{j-m}. \tag{11}$$

4. Proof of part (a). In this part, we are assuming,

$$N(x)|y(x)|^{\delta}/M(x) \to +\infty \quad \text{as } x \to +\infty \text{ for each } \delta \text{ in } (0, 1). \tag{12}$$

Thus it follows that for all sufficiently large x, the function $y(x)$ is either always positive or always negative. We can assume the positive case or otherwise we can apply the argument to $-y$. Hence since $M(x) \geqslant N(x)$, we can assume from (12) that

$$y(x) \to +\infty \quad \text{as } x \to +\infty. \tag{13}$$

We now assert that for any $\varepsilon > 0$, there exists a sequence $\{s_n\} \to +\infty$ such that,

$$|y'(s_n)| \leqslant (y(s_n))^{1+\varepsilon} \quad \text{for each } n. \tag{14}$$

To prove (14), we assume the contrary. Then the inequality $|y'| > y^{1+\varepsilon}$ would hold on an interval $(x_1, +\infty)$. Setting $g = y^{-\varepsilon}$, it would follow that on $(x_1, +\infty)$, we would have $|g'| > \varepsilon$ and hence either $g' > \varepsilon$ or $g' < -\varepsilon$. However, $g' > \varepsilon$ leads to $g \to +\infty$ as $x \to +\infty$ which contradicts (13).

Similarly, $g' < -\varepsilon$ leads to $g \to -\infty$ as $x \to +\infty$ which again contradicts (13), and thus (14) is proved.

We now assert that for any $\varepsilon > 0$, we have

$$|y'(x)| \leqslant y(x)^{1+\varepsilon} \quad \text{for all sufficiently large } x. \tag{15}$$

To prove (15), we choose $\delta > 0$ so small that $\delta < \min\{\varepsilon, 1\}$ and such that,

$$\delta < (p - (i + j))/(j + 1) \quad \text{if } f_{ij} \not\equiv 0 \text{ and } i + j < p. \tag{16}$$

It clearly suffices (in view of (13)) to show that,

$$|y'(x)| \leqslant y(x)^{1+\delta} \quad \text{for all sufficiently large } x. \tag{17}$$

We assume that (17) is false. In view of (14), it follows that there exists a sequence $\{t_n\} \to +\infty$ at which $|y'| = y^{1+\delta}$. For $i + j < p$, we consider the functions h_{ij} defined in (9). It easily follows from (2), (13) and our choice of δ in (16), that $|h_{ij}(t_n)| \leqslant M(t_n)(y(t_n))^{-\delta}$ for all sufficiently large n, and hence from (8), we obtain,

$$|\Phi(t_n)| \leqslant cM(t_n)(y(t_n))^{-\delta}, \tag{18}$$

for all sufficiently large n, where c is a positive constant. We now estimate $|\Lambda(t_n)|$ from below, where Λ is given by (10). If $m = 0$, then from (3) and (18), it would follow since $\Lambda = \Phi$, that $N(t_n)|y(t_n)|^{\delta}/M(t_n) \leqslant c$ for all sufficiently large n, which clearly contradicts (12). Hence $m \geqslant 1$. We consider the functions Ψ_j defined in (11) for $j = 0, 1, \ldots, m - 1$. Since $|y'| = y^{1+\delta}$ at the points t_n, it easily follows from (2), (3) and (13), that for $j = 0, 1, \ldots, m - 1$, we have $|\Psi_j| \leqslant (M/N)y^{-\delta}$ at the points t_n, and hence from (12), it follows that $|\Psi_j(t_n)| \leqslant (1/(m + 1))$ for all sufficiently large n. Thus from (10) and (3), we see that $|\Lambda| \geqslant Ny^{\delta m}/(m + 1)$ at t_n for all sufficiently large n, and hence in view of (13), it follows that $|\Lambda(t_n)| \geqslant N(t_n)$ for all sufficiently large n. This, together with (18) and the fact that $\Lambda = \Phi$, yields the same contradiction of (12) as we obtained in the case $m = 0$, and this proves (17) and hence (15).

For $i + j < p$, we can now estimate the functions $h_{ij}(x)$ defined in (9). We choose δ in (0, 1) satisfying (16). Now if $j = 0$, then in view of (2) and (13), we have $|h_{ij}(x)| \leqslant M(x)(y(x))^{-\delta}$ for all sufficiently large x. If $j > 0$, we can obviously write $h_{ij} = f_{ij}(y'/y^{1+\alpha})^j y^{-\delta}$ where $\alpha > \delta > 0$. Thus, in view of (2) and (15), we obtain, $|h_{ij}(x)| \leqslant M(x)(y(x))^{-\delta}$ for all sufficiently large x. Hence, from (8), it clearly follows that,

$$|\Phi(x)| \leqslant cM(x)(y(x))^{-\delta} \quad \text{for all sufficiently large } x, \tag{19}$$

where c is a constant.

We now estimate $\Lambda(x)$ which is given by (10). We observe first that we must have $m \geqslant 1$, for if $m = 0$, then from (3) we would have $|\Lambda(x)| \geqslant N(x)$ for all sufficiently large x, and this, together with (19) and the fact that $\Lambda = \Phi$ would clearly contradict (12). Hence, we must have $m \geqslant 1$ if there is a solution satisfying (12). For convenience, let us denote by B, the set of all $x \geqslant x_0$ for which $|y'(x)/y(x)| \geqslant U(x)$, where $U(x) = (m + 1)M(x)/N(x)$.

Since $U(x) \geqslant 1$, it follows easily from the definition of $\Psi_j(x)$ in (11) and the estimates in (2) and (3), that for $j = 0, 1, \ldots, m-1$,

$$|\Psi_j(x)| \leqslant (1/(m+1)) \quad \text{if } x \text{ belongs to } B. \tag{20}$$

Hence from the representation for $\Lambda(x)$ in (10), together with the estimate (3), we obtain,

$$|\Lambda(x)| \geqslant N(x)|y'(x)/y(x)|^m/(m+1) \quad \text{for } x \text{ in } B. \tag{21}$$

Thus, if x belongs to B, then since $m \geqslant 1$, it follows from (21) that $|\Lambda(x)| \geqslant M(x)$. But from (19) and (13), it follows that for all sufficiently large x, we have $|\Phi(x)| < M(x)$, and hence since $\Lambda = \Phi$, we must conclude that if x is sufficiently large, then x cannot belong to B. Hence there exists $x_2 \geqslant x_0$ such that,

$$|y'(x)/y(x)| < U(x) \quad \text{for all } x \geqslant x_2. \tag{22}$$

The conclusion (4) now follows easily and part (a) is thus proved.

5. Proof of part (b). First, let us define,

$$L(x) = \exp\left(U(x) + \int_{x_0}^{x} U(t)\,dt\right) \quad \text{for } x \geqslant x_0. \tag{23}$$

From our assumptions about $U(x)$ in this part, it easily follows that $L(x)$ is a positive, increasing, differentiable function such that,

$$L(x) \to +\infty \quad \text{as } x \to +\infty, \tag{24}$$

and such that for any $\varepsilon > 0$, we have

$$L' \leqslant L^{1+\varepsilon} \quad \text{for all sufficiently large } x. \tag{25}$$

In this part, we are assuming that our solution $y(x)$ has the property that for some δ in (0, 1), the function $N(x)|y(x)|^\delta/M(x)$ does not tend to $+\infty$ as $x \to +\infty$. Hence there exists a constant K and a sequence $\{\zeta_n\} \to +\infty$ such that $|y| \leqslant (KM/N)^{1/\delta}$ at the points ζ_n. Since $U = (m+1)M/N$, it easily follows from (23) that,

$$|y(\zeta_n)| < L(\zeta_n) \quad \text{for all sufficiently large } n. \tag{26}$$

We now assert that,

$$|y(x)| \leqslant L(x) \quad \text{for all sufficiently large } x. \tag{27}$$

To prove (27), we assume the contrary. Then there is a sequence $\{s_n\} \to +\infty$ such that,

$$|y(s_n)| > L(s_n) \quad \text{for all } n. \tag{28}$$

Let J denote the set of all $x > x_0$ for which $|y(x)| > L(x)$. In view of (26) and (28), clearly J contains a countable union of disjoint, finite open intervals (a_n, b_n) such that $\{b_n\} \to +\infty$ and $|y(x)| = L(x)$ if $x = a_n$ or $x = b_n$. Now since $y(x)$ is clearly of constant sign on each interval (a_n, b_n), it follows from Rolle's theorem (applied to $y(x)/L(x)$), that there is a sequence $\{r_n\} \to +\infty$ such that,

$$y'(r_n)/y(r_n) = L'(r_n)/L(r_n) \quad \text{and} \quad |y(r_n)| > L(r_n). \tag{29}$$

We now estimate the value at r_n of the functions h_{ij} introduced in (9) for $i + j < p$. In view of (29) and (2), clearly at the point r_n,

$$|h_{ij}| \leqslant M(L'/L)^j L^{i+j-p}. \tag{30}$$

If $j = 0$, then clearly (in view of (24)), $|h_{ij}| \leqslant ML^{-1}$ at r_n. Now choose a constant σ in (0, 1) so small that

$$\sigma < (p - (i + j))/(j + 1) \quad \text{if } f_{ij} \not\equiv 0 \text{ and } i + j < p. \tag{31}$$

Then, if $j > 0$, we can write the right hand side of (30) as $M(L'/L^{1+\alpha})^j L^{-\sigma}$ where $\alpha > \sigma > 0$, so in view of (25) and (30), we have $|h_{ij}(r_n)| \leqslant M(r_n)(L(r_n))^{-\sigma}$ for all sufficiently large n. Hence from (8), we obtain,

$$|\Phi(r_n)| \leqslant cM(r_n)(L(r_n))^{-\sigma} \quad \text{for all sufficiently large } n, \tag{32}$$

where c is a constant. We now consider the value at r_n of the functions Ψ_j introduced in (11) for $j = 0, 1, \ldots, m - 1$. Clearly, from (2), (3) and (29), we have,

$$|\Psi_j(r_n)| \leqslant (M(r_n)/N(r_n))(L'(r_n)/L(r_n))^{j-m}. \tag{33}$$

But from (23), $L'/L \geqslant U$ where $U = (m + 1)M/N$. Hence since $j < m$ in (33), it follows that $|\Psi_j(r_n)| \leqslant (1/(m + 1))$ for all n. It thus follows from (3), (10) and (29) that for all n,

$$|\Lambda(r_n)| \geqslant N(r_n)(L'(r_n)/L(r_n))^m/(m + 1). \tag{34}$$

If $m = 0$, then since $\Lambda = \Phi$, it would easily follow from (32) and (34) that for all sufficiently large n, we would have $L(r_n) \leqslant (cU(r_n))^{1/\sigma}$. However, this is clearly impossible since from (23) it easily follows that $L(x)/(U(x))^{1/\sigma}$ tends to $+\infty$ as $x \to +\infty$. On the other hand, if $m \geqslant 1$, then since $cL^{-\sigma} \to 0$ as $x \to +\infty$, it would follow from (32) and (34) that for all sufficiently large n, we have $(L'(r_n)/L(r_n))^m < U(r_n)$. However, this is clearly impossible since from (23), $L'/L \geqslant U$ and, of course, $m \geqslant 1$ and $U(x) \geqslant 1$ for all x. This proves (27) and hence part (b) is proved.

6. Remark. In part (b) of the theorem, we require that the monotone nondecreasing function $U(x)$ be differentiable and have the property that for any $\varepsilon > 0$, the ratio $U'/\exp(\varepsilon U)$ tends to zero as $x \to +\infty$. We mention here the simple fact that for *any* differentiable, unbounded, monotone nondecreasing function U on an interval $[x_0, +\infty)$, and any $\varepsilon > 0$, it is easy to see that the ratio $U'/\exp(\varepsilon U)$ tends to zero as x tends to $+\infty$ outside a possible exceptional set of finite measure. This follows easily since the convergence of the integral $\int_{x_0}^{+\infty} (U'(t)/U(t)^{1+\varepsilon})\, dt$ implies that for any $\varepsilon > 0$, the inequality $U' < U^{1+\varepsilon}$ holds on $[x_0, +\infty)$ with the possible exception of a set of finite measure.

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