**A CHARACTERIZATION OF M. W. WILSON’S CRITERION FOR NONNEGATIVE EXPANSIONS OF ORTHOGONAL POLYNOMIALS**

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**Abstract.** Given a nonnegative function $f(x)$, M. W. Wilson observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) < 0, \quad i \neq j,$$

(1)

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$, have an expansion

$$P_n(x) = \sum_{k=0}^n a_{kn}Q_k(x)$$

with nonnegative coefficients $a_{kn} > 0$ where $Q_n(x)$, $Q_n(0) = 1$, are orthogonal relative to $d\mu(x)$. Recently it was shown that (1) holds for $f(x) = x^c$, $0 < c < 1$. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

Let $d\mu(x)$ be a positive measure on $[0, \infty)$ with $\mu(x)$ having an infinite number of points of increase. Suppose also that $d\mu(x)$ has finite moments of all orders and $Q_n(x)$, $n = 0, 1, \ldots$, are the associated orthogonal polynomials normalized so that $Q_n(0) = 1$. Given a nonnegative function $f(x)$, M. W. Wilson [5] observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) < 0, \quad i \neq j,$$

(1)

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$ have an expansion

$$P_n(x) = \sum_{k=0}^n a_{kn}Q_k(x)$$

with nonnegative coefficients, $a_{kn} > 0$. Recently, W. Trench [3] showed that (1) holds for $f(x) = x^c$, $0 < c < 1$, thus establishing a conjecture made by Askey [1]. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

**Theorem.** Let $d\mu(x)$, $Q_n(x)$, $n = 0, 1, 2, \ldots$, and $f(x)$ be as described above. Then

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) < 0, \quad i \neq j,$$

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If and only if \( f'(x) \) is completely monotonic on \([0, \infty)\).

**Proof.** First we prove the necessity of this condition on \( f(x) \). Thus, in particular, we have that

\[
\int_0^\infty f(x)Q_n(x) \, d\mu(x) < 0, \quad n > 1.
\]  

(2)

Fix \( 0 < x_1 < \cdots < x_{n+1} \) and let \( d\nu(x) \) be any positive measure supported on \( x_1, \ldots, x_{n+1} \). Then by a straightforward limiting argument we conclude from (2) (which holds for all \( d\mu \) with \( \mu(x) \) having an infinite number of points of increase) that

\[
\int_0^\infty f(x)Q_n(x) \, d\nu(x) < 0.
\]

Let \( p(x) \) be the polynomial of degree \( n \) which interpolates \( f \) at \( x_1, \ldots, x_{n+1} \). Then \( f(x) = p(x) \) a.e. \( d\nu(x) \) and consequently

\[
\int_0^\infty f(x)Q_n(x) \, d\nu(x) = f[x_1, \ldots, x_{n+1}] \int_0^\infty x^n Q_n(x) \, d\nu(x)
\]

\[
= q_n^{-1} f[x_1, \ldots, x_{n+1}] \int_0^\infty Q_n^2(x) \, d\nu(x)
\]

where \( Q_n(x) = q_n x^n + \cdots \) and \( f[x_1, \ldots, x_{n+1}] \) is the divided difference of \( f(x) \) at \( x_1, \ldots, x_{n+1} \). Since \( Q_n(0) = 1 \), it follows that \( \text{sgn } q_n = (-1)^n \) and hence

\[
(-1)^{n+1} f[x_1, \ldots, x_{n+1}] > 0, \quad n > 1.
\]

Thus \( (-1)^n f^{(n+1)}(x) > 0 \) and hence according to Widder [3], \( f'(x) \) is completely monotonic on \([0, \infty)\).

Conversely, let \( f'(x) \) be completely monotonic. Then \( f(x) = O(x) \) as \( x \to \infty \) and so the integrals in (1) are finite. Choose an \( a > 0 \) and suppose \( Q_n(x; a), n = 0, 1, 2, \ldots, \) are the polynomials orthogonal to \( d\mu(x) \) on \([0, a] \)

\[
\int_0^a Q_i(x; a)Q_j(x; a) \, d\mu(x) = 0, \quad i \neq j,
\]

\[
Q_i(0; a) = 1.
\]

Then clearly \( \lim_{a \to \infty} Q_i(x; a) = Q_i(x), x > 0, \) uniformly on bounded intervals. Hence to prove (1) it is sufficient to demonstrate that

\[
\int_0^a f(x)Q_i(x; a)Q_j(x; a) \, d\mu(x) < 0, \quad i \neq j.
\]

For convenience we suppress the \( a \) when writing \( Q_i(x; a) \) and let it be understood that these polynomials are orthogonal with respect to \( d\mu(x) \) over \([0, a] \). \( Q_n(x) \) satisfies a three term recurrence relation

\[
-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x)
\]

\[
\mu_0 = 0, \mu_{n+1} > 0, \lambda_n > 0, n > 0,
\]

and since \( Q_n(0) = 1 \)
\[ \int_0^a Q_i(x) Q_j(x) \, d\mu(x) = \delta_{ij} \sigma_j^{-1} \]

where \( \sigma_0 = 1, \sigma_j = \lambda_0 \lambda_1 \cdots \lambda_{j-1}/\mu_1 \mu_2 \cdots \mu_j, j > 1. \)

Consequently, \( \hat{Q}_i = \pi_i^{1/2} Q_i \) are orthonormal and according to (3)

\[ A_{ij} = \int_0^a x \hat{Q}_i(x) \hat{Q}_j(x) \, d\mu(x) < 0, \quad i \neq j. \quad (4) \]

Now, fix a positive integer \( p \) and consider the \( p + 1 \times p + 1 \) section of the infinite matrix \( A \) defined above,

\[ \int_0^a x \hat{Q}_i(x) \hat{Q}_j(x) \, d\mu(x), \quad 0 < i, j < p. \]

\( A_p \) is a Stieltjes matrix, because it is obviously symmetric and positive definite and according to (4) has nonpositive off diagonal elements.

This matrix has an easily derived spectral decomposition. In fact, if \( 0 < x_1 < \cdots < x_{p+1} \) are the zeros of \( Q_{p+1}(x) \), then Gaussian quadrature for \( d\mu(x) \) on \([0, a]\) gives

\[ \int_0^a F(x) \, d\mu(x) = \sum_{k=1}^{p+1} w_k F(x_k), \quad F \in P_{2p+1}, \quad (5) \]

\( P_{2p+1} = \) polynomials of degree \( \leq 2p + 1 \). Using the Christoffel Darboux formula

\[ (x - y) \sum_{k=0}^n Q_k(x) Q_k(y) \pi_k = \lambda_n \pi_n (Q_n(x) Q_{n+1}(y) - Q_n(y) Q_{n+1}(x)) \]

a straightforward calculation shows that

\[ \sum_{k=1}^{p+1} Q_k(x_l) Q_k(x_m) \pi_k = \delta_{lm} w_l^{-1}, \quad l, m \leq p. \]

Hence the \( p + 1 \)-dimensional vectors \( u_l = (w_l^{1/2} \hat{Q}_0(x_l), \ldots, w_l^{1/2} \hat{Q}_p(x_l))^T \) are orthonormal,

\[ (u_l, u_m) = \delta_{lm}, \quad l, m \leq p. \]

Applying the Gaussian quadrature formula (5) to \( F(x) = x \hat{Q}_i(x) \hat{Q}_j(x) \) gives us the desired representation for \( A_p \),

\[ A_p = \sum_{l=1}^{p+1} x_l u_l u_l^T. \quad (6) \]

Now, we are ready to use the fact that \( f'(x) \) is completely monotonic. This property of \( f \) implies that

\[ f(x) = f(x_{p+1}) - \sum_{m=0}^{\infty} \frac{(-1)^{m+1} f^{(m)}(x_{p+1})}{m!} (x_{p+1} - x)^m \]

and thus

\[ f(A_p) = f(x_{p+1}) I - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} f^{(m)}(x_{p+1}) (x_{p+1} I - A_p)^m. \quad (7) \]
The matrix \( x_{p+1} I - A_p \) is positive definite, symmetric and has all nonnegative elements. Therefore (7) shows that \( f(A_p) \) is also a Stieltjes matrix (for a discussion of functions which preserve the class of Stieltjes matrices see [2]). Consequently, we have shown that for all \( p \)

\[
f(A_p)_{ij} < 0, \quad i \neq j, i, j < p.
\]

The spectral decomposition (6) of \( A_p \) gives us an alternate expression for \( f(A_p) \). Indeed (6) implies

\[
f(A_p) = \sum_{l=1}^{p+1} f(x_l) u_l u_l^T,
\]

or equivalently

\[
f(A_p)_{ij} = \sum_{l=1}^{p+1} f(x_l) w_l \hat{Q}_i(x_l) \hat{Q}_j(x_l).
\]

Letting \( p \to \infty \) and noting that the right hand side of (9) is the result of applying the Gaussian rule (5) to \( F = f\hat{Q}_i \hat{Q}_j \) we obtain

\[
0 \geq \lim_{p \to \infty} f(A_p)_{ij} = \int_0^a f(x) \hat{Q}_i(x) \hat{Q}_j(x) \, d\mu(x), \quad i \neq j.
\]

This assertion completes the proof of the theorem.

References


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