

A CHARACTERIZATION OF M. W. WILSON'S CRITERION FOR NONNEGATIVE EXPANSIONS OF ORTHOGONAL POLYNOMIALS

CHARLES A. MICCHELLI

ABSTRACT. Given a nonnegative function $f(x)$, M. W. Wilson observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x) d\mu(x) < 0, \quad i \neq j, \quad (1)$$

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$, have an expansion

$$P_n(x) = \sum_{k=0}^n a_{kn}Q_k(x)$$

with nonnegative coefficients $a_{kn} \geq 0$ where $Q_n(x)$, $Q_n(0) = 1$, are orthogonal relative to $d\mu(x)$. Recently it was shown that (1) holds for $f(x) = x^c$, $0 < c < 1$. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

Let $d\mu(x)$ be a positive measure on $[0, \infty)$ with $\mu(x)$ having an infinite number of points of increase. Suppose also that $d\mu(x)$ has finite moments of all orders and $Q_n(x)$, $n = 0, 1, \dots$, are the associated orthogonal polynomials normalized so that $Q_n(0) = 1$. Given a nonnegative function $f(x)$, M. W. Wilson [5] observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x) d\mu(x) < 0, \quad i \neq j, \quad (1)$$

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$ have an expansion

$$P_n(x) = \sum_{k=0}^n a_{kn}Q_k(x)$$

with nonnegative coefficients, $a_{kn} \geq 0$. Recently, W. Trench [3] showed that (1) holds for $f(x) = x^c$, $0 < c < 1$, thus establishing a conjecture made by Askey [1]. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

THEOREM. Let $d\mu(x)$, $Q_n(x)$, $n = 0, 1, 2, \dots$, and $f(x)$ be as described above. Then

$$\int_0^\infty f(x)Q_i(x)Q_j(x) d\mu(x) \leq 0, \quad i \neq j,$$

Received by the editors August 24, 1977.

AMS (MOS) subject classifications (1970). Primary 42A52.

if and only if $f'(x)$ is completely monotonic on $[0, \infty)$.

PROOF. First we prove the necessity of this condition on $f(x)$. Thus, in particular, we have that

$$\int_0^\infty f(x) Q_n(x) d\mu(x) \leq 0, \quad n \geq 1. \tag{2}$$

Fix $0 < x_1 < \dots < x_{n+1}$ and let $d\nu(x)$ be any positive measure supported on x_1, \dots, x_{n+1} . Then by a straightforward limiting argument we conclude from (2) (which holds for all $d\mu$ with $\mu(x)$ having an infinite number of points of increase) that

$$\int_0^\infty f(x) Q_n(x) d\nu(x) \leq 0.$$

Let $p(x)$ be the polynomial of degree n which interpolates f at x_1, \dots, x_{n+1} . Then $f(x) = p(x)$ a.e. $d\nu(x)$ and consequently

$$\begin{aligned} \int_0^\infty f(x) Q_n(x) d\nu(x) &= f[x_1, \dots, x_{n+1}] \int_0^\infty x^n Q_n(x) d\nu(x) \\ &= q_n^{-1} f[x_1, \dots, x_{n+1}] \int_0^\infty Q_n^2(x) d\nu(x) \end{aligned}$$

where $Q_n(x) = q_n x^n + \dots$ and $f[x_1, \dots, x_{n+1}]$ is the divided difference of $f(x)$ at x_1, \dots, x_{n+1} . Since $Q_n(0) = 1$, it follows that $\text{sgn } q_n = (-1)^n$ and hence

$$(-1)^{n+1} f[x_1, \dots, x_{n+1}] \geq 0, \quad n \geq 1.$$

Thus $(-1)^n f^{(n+1)}(x) \geq 0$ and hence according to Widder [3], $f'(x)$ is completely monotonic on $[0, \infty)$.

Conversely, let $f'(x)$ be completely monotonic. Then $f(x) = O(x)$ as $x \rightarrow \infty$ and so the integrals in (1) are finite. Choose an $a > 0$ and suppose $Q_n(x; a)$, $n = 0, 1, 2, \dots$, are the polynomials orthogonal to $d\mu(x)$ on $[0, a]$

$$\begin{aligned} \int_0^a Q_i(x; a) Q_j(x; a) d\mu(x) &= 0, \quad i \neq j, \\ Q_i(0; a) &= 1. \end{aligned}$$

Then clearly $\lim_{a \rightarrow \infty} Q_i(x; a) = Q_i(x)$, $x \geq 0$, uniformly on bounded intervals. Hence to prove (1) it is sufficient to demonstrate that

$$\int_0^a f(x) Q_i(x; a) Q_j(x; a) d\mu(x) \leq 0, \quad i \neq j.$$

For convenience we suppress the a when writing $Q_i(x; a)$ and let it be understood that these polynomials are orthogonal with respect to $d\mu(x)$ over $[0, a]$. $Q_n(x)$ satisfies a three term recurrence relation

$$\begin{aligned} -xQ_n(x) &= \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n) Q_n(x) + \lambda_n Q_{n+1}(x) \\ \mu_0 &= 0, \mu_{n+1} > 0, \lambda_n > 0, n \geq 0, \end{aligned} \tag{3}$$

and since $Q_n(0) = 1$

$$\int_0^a Q_i(x)Q_j(x) d\mu(x) = \delta_{ij}\pi_j^{-1}$$

where $\pi_0 = 1, \pi_j = \lambda_0\lambda_1 \cdots \lambda_{j-1}/\mu_1\mu_2 \cdots \mu_j, j \geq 1$.

Consequently, $\hat{Q}_i = \pi_i^{1/2}Q_i$ are orthonormal and according to (3)

$$A_{ij} := \int_0^a x\hat{Q}_i(x)\hat{Q}_j(x) d\mu(x) \leq 0, \quad i \neq j. \tag{4}$$

Now, fix a positive integer p and consider the $p + 1 \times p + 1$ section of the infinite matrix A defined above,

$$(A_p)_{ij} = \int_0^a x\hat{Q}_i(x)\hat{Q}_j(x) d\mu(x), \quad 0 \leq i, j \leq p.$$

A_p is a *Stieltjes* matrix, because it is obviously symmetric and positive definite and according to (4) has nonpositive off diagonal elements.

This matrix has an easily derived spectral decomposition. In fact, if $0 < x_1 < \cdots < x_{p+1}$ are the zeros of $Q_{p+1}(x)$, then Gaussian quadrature for $d\mu(x)$ on $[0, a]$ gives

$$\int_0^a F(x) d\mu(x) = \sum_{k=1}^{p+1} w_k F(x_k), \quad F \in P_{2p+1}, \tag{5}$$

P_{2p+1} = polynomials of degree $\leq 2p + 1$. Using the Christoffel Darboux formula

$$(x - y) \sum_{k=0}^n Q_k(x)Q_k(y)\pi_k = \lambda_n\pi_n(Q_n(x)Q_{n+1}(y) - Q_n(y)Q_{n+1}(x))$$

a straightforward calculation shows that

$$\sum_{k=1}^{p+1} Q_k(x_l)Q_k(x_m)\pi_k = \delta_{lm}w_l^{-1}, \quad l, m \leq p.$$

Hence the $p + 1$ -dimensional vectors $u_l = (w_l^{1/2}\hat{Q}_0(x_l), \dots, w_l^{1/2}\hat{Q}_p(x_l))^T$ are orthonormal,

$$(u_l, u_m) = \delta_{lm}, \quad l, m \leq p.$$

Applying the Gaussian quadrature formula (5) to $F(x) = x\hat{Q}_i(x)\hat{Q}_j(x)$ gives us the desired representation for A_p ,

$$A_p = \sum_{l=1}^{p+1} x_l u_l u_l^T. \tag{6}$$

Now, we are ready to use the fact that $f'(x)$ is completely monotonic. This property of f implies that

$$f(x) = f(x_{p+1}) - \sum_{m=0}^{\infty} \frac{(-1)^{m+1}f^{(m)}(x_{p+1})}{m!} (x_{p+1} - x)^m$$

and thus

$$f(A_p) = f(x_{p+1})I - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} f^{(m)}(x_{p+1})(x_{p+1}I - A_p)^m. \tag{7}$$

The matrix $x_{p+1}I - A_p$ is positive definite, symmetric and has all *nonnegative* elements. Therefore (7) shows that $f(A_p)$ is also a Stieltjes matrix (for a discussion of functions which preserve the class of Stieltjes matrices see [2]). Consequently, we have shown that for all p

$$f(A_p)_{ij} \leq 0, \quad i \neq j, i, j \leq p. \quad (8)$$

The spectral decomposition (6) of A_p gives us an alternate expression for $f(A_p)$. Indeed (6) implies

$$f(A_p) = \sum_{l=1}^{p+1} f(x_l) u_l u_l^T$$

or equivalently

$$f(A_p)_{ij} = \sum_{l=1}^{p+1} f(x_l) w_l \hat{Q}_i(x_l) \hat{Q}_j(x_l). \quad (9)$$

Letting $p \rightarrow \infty$ and noting that the right hand side of (9) is the result of applying the Gaussian rule (5) to $F = f\hat{Q}_i\hat{Q}_j$ we obtain

$$0 \geq \lim_{p \rightarrow \infty} f(A_p)_{ij} = \int_0^a f(x) \hat{Q}_i(x) \hat{Q}_j(x) d\mu(x), \quad i \neq j.$$

This assertion completes the proof of the theorem.

REFERENCES

1. R. Askey, *Orthogonal expansions with positive coefficients*, Proc. Amer. Math. Soc. **16** (1965), 1191–1194.
2. C. A. Micchelli and R. A. Willoughby, *On functions which preserve the class of Stieltjes matrices*, Linear Algebra and Appl. (to appear).
3. William F. Trench, *Proof of a conjecture of Askey on orthogonal expansions with positive coefficients*, Bull. Amer. Math. Soc. **81** (1975), 954–956.
4. D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, N.J., 1946.
5. M. W. Wilson, *Nonnegative expansions of polynomials*, Proc. Amer. Math. Soc. **24** (1970), 100–102.

IBM T. J. WATSON RESEARCH CENTER, YORKTOWN HEIGHTS, NEW YORK 10598