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A CHARACTERIZATION OF M. W. WILSON'S CRITERION FOR NONNEGATIVE EXPANSIONS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. Given a nonnegative function $f(x)$, M. W. Wilson observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) \leqslant 0, \qquad i \neq j, \tag{1}$$

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$, have an expansion

$$P_n(x) = \sum_{k=0}^{n} a_{kn}Q_k(x)$$

with nonnegative coefficients $a_{kn} \geqslant 0$ where $Q_n(x)$, $Q_n(0) = 1$, are orthogonal relative to $d\mu(x)$. Recently it was shown that (1) holds for $f(x) = x^c$, $0 < c < 1$. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

Let $d\mu(x)$ be a positive measure on $[0, \infty)$ with $\mu(x)$ having an infinite number of points of increase. Suppose also that $d\mu(x)$ has finite moments of all orders and $Q_n(x)$, $n = 0, 1, \ldots$, are the associated orthogonal polynomials normalized so that $Q_n(0) = 1$. Given a nonnegative function $f(x)$, M. W. Wilson [5] observed that if

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) \leqslant 0, \qquad i \neq j, \tag{1}$$

then the polynomials $P_n(x)$, $P_n(0) = 1$, orthogonal relative to $f(x)d\mu(x)$ have an expansion

$$P_n(x) = \sum_{k=0}^{n} a_{kn}Q_k(x)$$

with nonnegative coefficients, $a_{kn} \geqslant 0$. Recently, W. Trench [3] showed that (1) holds for $f(x) = x^c$, $0 < c < 1$, thus establishing a conjecture made by Askey [1]. In this paper we characterize those functions $f(x)$ for which (1) is valid for all positive measures $d\mu(x)$.

THEOREM. *Let $d\mu(x)$, $Q_n(x)$, $n = 0, 1, 2, \ldots,$ and $f(x)$ be as described above. Then*

$$\int_0^\infty f(x)Q_i(x)Q_j(x)\,d\mu(x) \leqslant 0, \qquad i \neq j,$$

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if and only if $f'(x)$ *is completely monotonic on* $[0, \infty)$.

PROOF. First we prove the necessity of this condition on $f(x)$. Thus, in particular, we have that

$$\int_0^\infty f(x)Q_n(x)\,d\mu(x) \leqslant 0, \qquad n \geqslant 1. \tag{2}$$

Fix $0 < x_1 < \cdots < x_{n+1}$ and let $d\nu(x)$ be any positive measure supported on $x_1, \ldots, x_{n+1}$. Then by a straightforward limiting argument we conclude from (2) (which holds for all $d\mu$ with $\mu(x)$ having an infinite number of points of increase) that

$$\int_0^\infty f(x)Q_n(x)\,d\nu(x) \leqslant 0.$$

Let $p(x)$ be the polynomial of degree n which interpolates f at $x_1, \ldots, x_{n+1}$. Then $f(x) = p(x)$ a.e. $d\nu(x)$ and consequently

$$\int_0^\infty f(x)Q_n(x)\,d\nu(x) = f[x_1, \ldots, x_{n+1}]\int_0^\infty x^n Q_n(x)\,d\nu(x)$$
$$= q_n^{-1}f[x_1, \ldots, x_{n+1}]\int_0^\infty Q_n^2(x)\,d\nu(x)$$

where $Q_n(x) = q_n x^n + \cdots$ and $f[x_1, \ldots, x_{n+1}]$ is the divided difference of $f(x)$ at $x_1, \ldots, x_{n+1}$. Since $Q_n(0) = 1$, it follows that $\operatorname{sgn} q_n = (-1)^n$ and hence

$$(-1)^{n+1}f[x_1, \ldots, x_{n+1}] \geqslant 0, \qquad n \geqslant 1.$$

Thus $(-1)^n f^{(n+1)}(x) \geqslant 0$ and hence according to Widder [3], $f'(x)$ is completely monotonic on $[0, \infty)$.

Conversley, let $f'(x)$ be completely monotonic. Then $f(x) = O(x)$ as $x \to \infty$ and so the integrals in (1) are finite. Choose an $a > 0$ and suppose $Q_n(x; a)$, $n = 0, 1, 2, \ldots$, are the polynomials orthogonal to $d\mu(x)$ on $[0, a]$

$$\int_0^a Q_i(x; a)Q_j(x; a)\,d\mu(x) = 0, \qquad i \neq j,$$
$$Q_i(0; a) = 1.$$

Then clearly $\lim_{a\to\infty} Q_i(x; a) = Q_i(x)$, $x \geqslant 0$, uniformly on bounded intervals. Hence to prove (1) it is sufficient to demonstrate that

$$\int_0^a f(x)Q_i(x; a)Q_j(x; a)\,d\mu(x) \leqslant 0, \qquad i \neq j.$$

For convenience we suppress the a when writing $Q_i(x; a)$ and let it be understood that these polynomials are orthogonal with respect to $d\mu(x)$ over $[0, a]$. $Q_n(x)$ satisfies a three term recurrence relation

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x)$$
$$\mu_0 = 0,\ \mu_{n+1} > 0,\ \lambda_n > 0,\ n \geqslant 0, \tag{3}$$

and since $Q_n(0) = 1$

$$\int_0^a Q_i(x)Q_j(x)\,d\mu(x) = \delta_{ij}\pi_j^{-1}$$

where $\pi_0 = 1$, $\pi_j = \lambda_0\lambda_1 \cdots \lambda_{j-1}/\mu_1\mu_2 \cdots \mu_j, j \geq 1$.

Consequently, $\hat{Q}_i = \pi_i^{1/2}Q_i$ are orthonormal and according to (3)

$$A_{ij} := \int_0^a x\hat{Q}_i(x)\hat{Q}_j(x)\,d\mu(x) \leq 0, \qquad i \neq j. \tag{4}$$

Now, fix a positive integer p and consider the $p + 1 \times p + 1$ *section* of the infinite matrix A defined above,

$$(A_p)_{ij} = \int_0^a x\hat{Q}_i(x)\hat{Q}_j(x)\,d\mu(x), \qquad 0 \leq i,j \leq p.$$

A_p is a *Stieltjes* matrix, because it is obviously symmetric and positive definite and according to (4) has nonpositive off diagonal elements.

This matrix has an easily derived spectral decomposition. In fact, if $0 < x_1 < \cdots < x_{p+1}$ are the zeros of $Q_{p+1}(x)$, then Gaussian quadrature for $d\mu(x)$ on $[0, a]$ gives

$$\int_0^a F(x)\,d\mu(x) = \sum_{k=1}^{p+1} w_k F(x_k), \qquad F \in P_{2p+1}, \tag{5}$$

P_{2p+1} = polynomials of degree $\leq 2p + 1$. Using the Christoffel Darboux formula

$$(x-y)\sum_{k=0}^{n} Q_k(x)Q_k(y)\pi_k = \lambda_n\pi_n\big(Q_n(x)Q_{n+1}(y) - Q_n(y)Q_{n+1}(x)\big)$$

a straightforward calculation shows that

$$\sum_{k=1}^{p+1} Q_k(x_l)Q_k(x_m)\pi_k = \delta_{lm}w_l^{-1}, \qquad l, m \leq p.$$

Hence the $p + 1$-dimensional vectors $u_l = (w_l^{1/2}\hat{Q}_0(x_l), \ldots, w_l^{1/2}\hat{Q}_p(x_l))^T$ are orthonormal,

$$(u_l, u_m) = \delta_{lm}, \qquad l, m \leq p.$$

Applying the Gaussian quadrature formula (5) to $F(x) = x\hat{Q}_i(x)\hat{Q}_j(x)$ gives us the desired representation for A_p,

$$A_p = \sum_{l=1}^{p+1} x_l u_l u_l^T. \tag{6}$$

Now, we are ready to use the fact that $f'(x)$ is completely monotonic. This property of f implies that

$$f(x) = f(x_{p+1}) - \sum_{m=0}^{\infty} \frac{(-1)^{m+1}f^{(m)}(x_{p+1})}{m!}(x_{p+1} - x)^m$$

and thus

$$f(A_p) = f(x_{p+1})I - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} f^{(m)}(x_{p+1})(x_{p+1}I - A_p)^m. \tag{7}$$

The matrix $x_{p+1}I - A_p$ is positive definite, symmetric and has all *nonnegative* elements. Therefore (7) shows that $f(A_p)$ is also a Stieltjes matrix (for a discussion of functions which preserve the class of Stieltjes matrices see [2]). Consequently, we have shown that for all p

$$f(A_p)_{ij} \leqslant 0, \qquad i \neq j, i, j \leqslant p. \tag{8}$$

The spectral decomposition (6) of A_p gives us an alternate expression for $f(A_p)$. Indeed (6) implies

$$f(A_p) = \sum_{l=1}^{p+1} f(x_l) u_l u_l^T$$

or equivalently

$$f(A_p)_{ij} = \sum_{l=1}^{p+1} f(x_l) w_l \hat{Q}_i(x_l) \hat{Q}_j(x_l). \tag{9}$$

Letting $p \to \infty$ and noting that the right hand side of (9) is the result of applying the Gaussian rule (5) to $F = f\hat{Q}_i\hat{Q}_j$ we obtain

$$0 \geqslant \lim_{p \to \infty} f(A_p)_{ij} = \int_0^a f(x) \hat{Q}_i(x) \hat{Q}_j(x)\, d\mu(x), \qquad i \neq j.$$

This assertion completes the proof of the theorem.

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