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EXTENSION OF PRIVALOFF'S THEOREM TO ULTRASPHERICAL EXPANSIONS[1]

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ABSTRACT. Privaloff's theorem which relates the boundary continuity of $\mathrm{Re}(f)$ to that of f, f analytic on a disk, is extended to ultraspherical expansions.

Let $F(\theta) = \sum_{n=0}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$. Privaloff's theorem [**8**], [**1**, p. 99] states that if F satisfies a Lipschitz condition of order α, $0 < \alpha < 1$, i.e. $|F(\theta_1) - F(\theta_2)| \leqslant K|\theta_1 - \theta_2|^{\alpha}$, $-\pi \leqslant \theta_1, \theta_2 \leqslant \pi$, then the conjugate

$$G(\theta) = \sum_{n=1}^{\infty} (-b_n \cos n\theta + a_n \sin n\theta)$$

also satisfies a Lipschitz condition of the same order α. Thus if $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in the open disk $|z| < 1$, and u is continuous on the closed disk $|z| \leqslant 1$, with $U(\theta) = u(1, \theta)$ satisfying a Lipschitz condition of order α, it follows that f is continuous on the closed disk $|z| \leqslant 1$ and $F(\theta) = f(e^{i\theta})$ also satisfies a Lipschitz condition of order α. Thus by the result of Hardy-Littlewood [**7**, p. 427], f satisfies a complex Lipschitz condition of order α on the closed disk, i.e. $|f(z_1) - f(z_2)| \leqslant K|z_1 - z_2|^{\alpha}$, $|z_1|, |z_2| \leqslant 1$. This later result is also often taken as a statement of Privaloff's theorem (e.g. see [**2**, p. 380]).

In this paper we extend Privaloff's result to ultraspherical expansions $\sum_{n=0}^{\infty} a_n C_n^{\mu}(\cos\theta)$, C_n^{μ} being the Gegenbauer polynomial of degree n and index μ. Functions

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^{\mu}(\cos\theta) \tag{1}$$

appear as solutions of the partial differential equation

$$u_{xx} + u_{yy} + (2\mu/y)u_y = 0, \tag{2}$$

and are often called generalized axisymmetric potentials [**11**]. In the study of this equation R. P. Gilbert [**4**, pp. 165–174] has introduced an integral operator which transforms analytic functions of a single complex variable to solutions (1). Expressing u in rectangular coordinates, his A_{μ} operator is given by

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$$u(x,y) = A_\mu(f)$$
$$= \frac{2^{1-2\mu}\Gamma(2\mu)}{\Gamma^2(\mu)} \int_0^\pi f(x+iy\cos t)(\sin t)^{2\mu-1}\,dt, \tag{3}$$

and with u expressed in polar coordinates the integral representation for the inverse transform is

$$f(z) = A_\mu^{-1}(u)$$
$$= \int_0^\pi u(r,t)K(z/r, \cos t)\sin t\,dt, \qquad |z| < r, \tag{4}$$

where

$$K(\sigma,\xi) = \frac{\mu\Gamma(2\mu)}{2^{2\mu-1}\Gamma^2(\mu+1/2)} \frac{(1-\xi^2)^{\mu-1/2}(1-\sigma^2)}{(1-2\xi\sigma+\sigma^2)^{\mu+1}}.$$

When $\mu = 0$ equation (2) becomes Laplace's equation in two dimensions. For this reason it is convenient to normalize the A_μ operator as follows: let

$$L_\mu = \frac{2^{2\mu-1}}{\mu\Gamma(2\mu)} A_\mu, \qquad \mu > 0.$$

Then since

$$\lim_{\mu\to 0} \frac{C_n^\mu(x)}{\mu} = \frac{2}{n} T_n(x)$$

where T_n is the Tchebychev polynomial of degree n [**11**, p. 178], we have

$$\lim_{\mu\to 0} L_\mu(z^n) = \lim_{\mu\to 0} \frac{2^{2\mu-1}\Gamma(n+1)}{\mu\Gamma(n+2\mu)} r^n C_n^\mu(\cos\theta)$$
$$= r^n \cos n\theta = Re(z^n).$$

Thus when considering functions having real coefficients the operation of "taking the real part" appears as a limiting case of the L_μ operator when $\mu \to 0$. We accordingly define $L_\mu = Re$ for $\mu = 0$.

If $f(z) = \sum_{n=0}^\infty a_n z^n$, then for $\mu > 0$

$$u(r,\theta) = L_\mu(f)$$
$$= \mu^{-1}2^{2\mu-1} \sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+2\mu)} a_n r^n C_n^\mu(\cos\theta).$$

Each series has radius of convergence R, where $1/R = \limsup |a_n|^{1/n}$. Further, the singularities of u and its "L_μ associate" f on $|z| = R$ coincide in complex conjugate pairs (for discussions of singularities see [**3**], [**5**], [**9**]). We now consider the relationship of the continuity properties of u and f on the boundary circle.

THEOREM 1. *Let $f(z)$ be analytic in the open disk $|z| < 1$, and $u = L_\mu(f)$, $0 \leqslant \mu < 1$. If u is continuous on the closed disk $|z| \leqslant 1$, and on $|z| = 1$ satisfies* *the Lipschitz condition of order* α

$$|u(1,\theta) - u(1,\theta')| \leqslant K|\theta - \theta'|^\alpha, \qquad 0 \leqslant \mu < \alpha < 1,$$

then f is continuous on the closed disk $|z| \leqslant 1$. Further, on the circle $|z| = 1$, $F(\theta) = f(e^{i\theta})$ satisfies a Lipschitz condition of order $\alpha - \mu$.

Proof. In the case where $\mu = 0$ the above reduces to the statement of Privaloff's theorem. Let $\mu > 0$. Since u is continuous on $|z| \leqslant 1$ we may choose $r = 1$ in (4) and compute $f'(z)$, $|z| < 1$, by carrying out differentiation under the integral. Let $z = re^{i\theta}$. Noting that $d/dz\, L_\mu^{-1}(1) = 0$, we obtain

$$f'(z) = \frac{\mu^2\Gamma^2(2\mu)}{4^{2\mu-1}\Gamma^2(\mu + 1/2)} \int_0^\pi \left[u(1,t) - u(1,\theta)\right] \times \frac{d}{dz}\left[\frac{1 - z^2}{(1 - 2z\cos t + z^2)^{\mu+1}}\right](\sin t)^{2\mu}\, dt.$$

Absorbing all constants into a factor C preceding the integral, this yields

$$|f'(z)| \leqslant C\int_0^\pi \frac{|u(1,t) - u(1,\theta)|}{|(z - e^{it})(z - e^{-it})|^{\mu+2}}\, dt,$$

and since u is an even function of t the above is

$$\begin{aligned}
&\leqslant C_1\int_{-\pi}^{\pi} |u(1,t) - u(1,\theta)|\, |z - e^{it}|^{-\mu-2}\, dt \\
&= C_1\int_{-\pi}^{\pi} |u(\theta - \phi) - u(\theta)|\, |1 - re^{i\phi}|^{-\mu-2}\, d\phi \\
&\leqslant C_2\int_{-\pi}^{\pi} \frac{|\phi|^\alpha\, d\phi}{|1 - 2r\cos\phi + r^2|^{\mu/2+1}} \\
&= \frac{C_2}{(1-r)^{\mu+2}}\int_{-\pi}^{\pi} \frac{|\phi|^\alpha\, d\phi}{\left[1 + 4r(1-r)^{-2}(\sin\phi/2)^2\right]^{\mu/2+1}} \\
&\leqslant \frac{2C_2}{(1-r)^{\mu+2}}\int_0^\infty \frac{\phi^\alpha\, d\phi}{\left[1 + 4r(1-r)^{-2}(\phi/\pi)^2\right]^{\mu/2+1}} \\
&= \frac{C_3}{(1-r)^{\mu+1-\alpha}\, r^{(\alpha+1)/2}}\int_0^\infty \frac{x^\alpha\, dx}{(1 + x^2)^{\mu/2+1}}.
\end{aligned}$$

Since the integral is convergent, this implies

$$|f'(z)| = O\left((1-r)^{\alpha-\mu-1}\right), \quad \text{as } r \to 1.$$

Thus by the result of Hardy-Littlewood [7, pp. 426–427], $f(z)$ is continuous on the closed disk $|z| \leqslant 1$ and $F(\theta) = f(e^{i\theta})$ satisfies a Lipschitz condition of order $\alpha - \mu$.

Corollary. *Under the hypothesis of the previous theorem, f satisfies a complex Lipschitz condition of order $\alpha - \mu$ on the closed disk $|z| \leqslant 1$, i.e.*

$$|f(z_1) - f(z_2)| \leqslant C|z_1 - z_2|^{\alpha-\mu}, \quad for\ |z_1|, |z_2| < 1.$$

The preceding theorem allows us to relate continuity properties of trigonometric series $\sum_{n=0}^{\infty} a_n e^{inx}$ and ultraspherical expansions

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+2\mu)} a_n C_n^{\mu}(x), \qquad x \in [-1, 1], \tag{5}$$

where $0 < \mu < 1$. For example, since $C_n^{1/2}(x) = P_n(x)$ where P_n are the Legendre polynomials. on letting $\mu = 1/2$ in (5) we have the following result: *If $\sum_{n=0}^{\infty} a_n P_n(x)$ satisfies a Lipschitz condition of order $\alpha > 1/2$ on $[-1, 1]$, then $\sum_{m=0}^{\infty} a_n e^{inx}$ satisfies a Lipschitz condition of order $\alpha - 1/2$ on $[-\pi, \pi]$.*

We next obtain conditions on u which insure its L_μ associate is bounded.

THEOREM 2. *Let $f(z)$ be analytic in the open disk $|z| < 1$, and $u = L_\mu(f)$. If u is continuous on the closed disk $|z| \leqslant 1$ and*

$$|u(1, \theta) - u(1, \theta')| \leqslant \lambda(|\theta - \theta'|)$$

where $\int_0^{\pi/2} t^{-\mu-1} \lambda(t)\, dt$ converges, then $f(z)$ is bounded in the disk $|z| < 1$.

PROOF. The theorem for $\mu = 0$ is a classical function theoretic result (e.g. see [6, p. 415]). Let $\mu > 0$, and $z = re^{i\theta}$. Using (4) we have

$$f(z) = \frac{\mu^2\Gamma^2(2\mu)}{4^{2\mu-1}\Gamma^2(\mu+1/2)}$$
$$\times \int_0^{\pi} [u(1, t) - u(1, \theta)] \frac{1 - z^2}{(1 - 2z\cos t + z^2)^{\mu+1}} (\sin t)^{2\mu}\, dt$$
$$+ \beta_\mu u(1, \theta)$$

where $\beta_\mu = L_\mu^{-1}(1)$. Then arguing as in Theorem 1 yields

$$\begin{aligned} |f(z)| &\leqslant C \int_{-\pi}^{\pi} \frac{|u(1, t) - u(1, \theta)|}{|z - e^{it}|^{\mu+1}}\, dt + K \\ &\leqslant C \int_{-\pi}^{\pi} \frac{|u(1, \theta - \phi) - u(1, \theta)|}{|1 - re^{i\phi}|^{\mu+1}}\, d\phi + K \\ &< C \int_{-\pi}^{\pi} \frac{\lambda(|\phi|)\, d\phi}{|1 - re^{i\phi}|^{\mu+1}} + K \\ &= 2C \int_0^{\pi} \frac{\lambda(\phi)\, d\phi}{|1 - re^{i\phi}|^{\mu+1}} + K. \end{aligned}$$

The integral from $\pi/2$ to π remains bounded for all $|z| = r < 1$. Further, for $\phi \in [0, \pi/2]$, $|1 - re^{i\phi}| \geqslant |r \sin \phi| \geqslant r\phi$. Hence the above yields

$$|f(z)| \leqslant \frac{C_1}{r^{\mu+1}} \int_0^{\pi/2} \frac{\lambda(\phi)}{\phi^{\mu+1}}\, d\phi + K_1$$

where C_1, K_1 are constants. Thus as $|z| \to 1$, f remains bounded.

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