PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

ON THE DUALITY BETWEEN ASPLUND SPACES AND SPACES WITH THE RADON-NIKODÝM PROPERTY

FRANCIS SULLIVAN

ABSTRACT. A simple proof of the following result of Stegall is given. Let E be a real Banach space, then E is an Asplund space if E^* has the Radon-Nikodým property.

Let E be a real Banach space. E is said to be an Asplund space if every continuous convex real-valued function on an open convex subset of E is Fréchet differentiable at all points of a dense G_δ subset of its domain. This class of spaces was introduced by E. Asplund and called "strong differentiability spaces" by him. Asplund showed [2] that every weak* compact convex subset of E^* is weak* dentable. If A is a subset of a Banach space and f is a norm-1 linear functional then for each $\alpha > 0$, f determines a *slice* of A, i.e. the set $S(f, \alpha, A) = \{a \in A | f(a) \geqslant \sup f[A] - \alpha\}$. The set A is *dentable* if it has small slices, i.e. for each $\varepsilon > 0$ there is an f and an $\alpha > 0$ with diam $S(f, \alpha, A) < \varepsilon$. For a subset of a dual space E^*, weak* dentability means that the small slices are determined by elements of E.

Namioka and Phelps [4] proved that weak* dentability of weak* compact convex subsets of the dual characterizes Asplund spaces and in addition showed that *if E is an Asplund space then E^* has the Radon-Nikodým property*. We shall take the result of Stegall [6] as the definition of the Radon-Nikodým property for a dual space–namely E^* has the Radon-Nikodým property if and only if every separable subspace of E has separable dual. The original definition is, of course, in terms of integrals and there are fascinating connections between vector valued measures, extreme point phenomena and separability for dual spaces. An excellent source of information is the monograph of Diestel and Uhl [3].

Stegall [5] showed that E is an Asplund space if E^* has the Radon-Nikodým property, and an elegant simplification of Stegall's proof has been given by Namioka [3]. Stegall's result has recently been proved in quite a different way by Anantharaman, Lewis and Whitfield [1]. In the present note we give a proof which uses an essential idea of Anantharaman, Lewis and Whitfield but is much simpler in detail.

THEOREM. *If E^* has the Radon-Nikodým property then E is an Asplund space.*

Received by the editors December 20, 1977 and, in revised form, February 17, 1978.
AMS (MOS) subject classifications (1970). Primary 46B99.

PROOF. If E is not an Asplund space, then from the result of Namioka and Phelps, there is a weak* compact convex set $C \subseteq E^*$ which is not weak* dentable. If B^* is the unit ball of E^* then the weak* closed symmetric convex body $D \equiv B^* + (C - C)$ is also not weak* dentable. Hence we may assume that B^* is not weak* dentable. Using the definition of weak* dentability it is immediate that this means there is an $\varepsilon > 0$ such that for all $\|x\| = 1$ there are sequences (f_k), $(g_k) \subseteq B^*$ with $f_k(x)$, $g_k(x) \to 1$ while $(f_k - g_k)(y_k) > \varepsilon$ for some $\|y_k\| = 1$.

Let E_1 be any separable subspace of E and choose (x_n^1) to be a sequence dense in the unit sphere of E_1. From the above for each x_n^1 there are functionals $\{f_{nk}^1\}$, $\{g_{nk}^1\} \subseteq B^*$ with $f_{nk}^1(x_n^1)$, $g_{nk}^1(x_n^1) \to 1$ and norm-1 vectors y_{nk}^1 in E with $(f_{nk}^1 - g_{nk}^1)(y_{nk}^1) > \varepsilon$.

Let $E_2 \equiv \text{cl span}[E_1 \cup \{y_{nk}^1\}]$ and repeat the procedure to obtain separable subspaces $E_1 \subset E_2 \subset \cdots \subset E_l \subset E_{l+1} \subset \cdots$. Define $E_0 = \text{cl}(\bigcup E_l)$. We shall show that the unit ball of E_0^* is not weak* dentable. Since E^* has the Radon-Nikodým property E_0^* is separable [6] which, using [2] (or, more simply, the proof of [4, Corollary 10]), gives a contradiction.

Since $\bigcup E_l$ is dense in E_0, for any positive sequence $\delta_m \searrow 0$ and any $x_0 \in E_0$, $\|x_0\| = 1$, there is a norm-1 sequence $\{x_m\}$, where $x_m \equiv x_n^l$ for some l and n, such that $\|x_0 - x_m\| < \delta_m/2$. From the construction there are functionals $f_m \equiv f_{nk}^l$ and $g_m \equiv g_{nk}^l$ in B^* where k is chosen large enough that $f_m(x_m) \equiv f_{nk}^l(x_n^l) > 1 - \delta_m/2$ and $g_m(x_m) > 1 - \delta_m/2$. Thus $f_m(x_0) = f_m(x_0 - x_m) + f_m(x_m) > 1 - \delta_m$ and likewise $g_m(x_0) > 1 - \delta_m$, while for all m, $\|(f_m - g_m)|_{E_0}\| \geqslant (f_m - g_m)(y_{nk}^l) > \varepsilon$. Q.E.D.

REFERENCES

1. R. Anantharaman, T. Lewis and J. H. M. Whitfield, *Smoothability, strong smoothability and dentability in Banach spaces* (preprint).

2. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. **121** (1968), 31–47.

3. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R. I., 1977.

4. I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

5. C. Stegall, *The duality between Asplund spaces and spaces with the Radon-Nikodým property*, Israel J. Math. (to appear).

6. ______, *The Radon-Nikodým property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223.

DEPARTMENT OF MATHEMATICS, THE CATHOLIC UNIVERSITY OF AMERICA, WASHINGTON, D. C. 20064