

BOUNDARY REPRESENTATIONS AND TENSOR PRODUCTS OF C^* -ALGEBRAS¹

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ABSTRACT. Let A and B be unital, generating linear subspaces of C^* -algebras \mathcal{A} and \mathcal{B} , respectively. If either \mathcal{A} or \mathcal{B} is a GCR algebra, then the set of boundary representations for $A \otimes B$ can be identified with the Cartesian product of the boundary representations for A with the boundary representations for B .

Let \mathcal{A} be a unital C^* -algebra and let A be a linear subspace of \mathcal{A} which contains the unit, 1, and which generates \mathcal{A} as a C^* -algebra. An irreducible representation π of \mathcal{A} is said to be a *boundary representation* for A if π is the only completely positive extension to \mathcal{A} of the restriction $\pi|_A$. (A linear mapping ϕ on \mathcal{A} is *completely positive* if $\phi \otimes 1_n$ is positive for all $n = 1, 2, \dots$, where 1_n is the identity mapping on the algebra of $n \times n$ complex matrices.) Boundary representations were introduced by Arveson in [1], where he demonstrated the rôle played by boundary representations in determining the extent to which the order and norm structure on a unital linear space, A , of operators determines $C^*(A)$, the C^* -algebra generated by A .

If \mathcal{A} is a commutative C^* -algebra, a boundary representation is essentially just a point in the Choquet boundary of A . If \mathcal{A} is commutative, then $\mathcal{A} = C(X)$ for some compact Hausdorff space X and each irreducible representation π of \mathcal{A} is just point evaluation at some point $x \in X$. Every unital positive linear mapping on \mathcal{A} is completely positive and is also just integration with respect to some probability measure μ on X . The assertion that π is the only completely positive extension to \mathcal{A} of $\pi|_A$ becomes the assertion that point mass at x is the only probability measure μ on X for which

$$\int f(y)d\mu(y) = f(x),$$

for all $f \in A$. But this just says that x lies in the Choquet boundary for A . See [4] for more details.

If A is a unital, generating subspace of \mathcal{A} , let $\text{bd}(A)$ denote the set of boundary representations for A . Suppose that B is a unital, generating subspace of a second C^* -algebra \mathcal{B} . Let $\mathcal{A} \otimes \mathcal{B}$ denote the algebraic tensor

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product of \mathcal{A} and \mathcal{B} and let $\mathcal{A} \otimes_\gamma \mathcal{B}$ denote the closure of $\mathcal{A} \otimes \mathcal{B}$ when the latter is provided with the C^* -cross norm γ . The algebraic tensor product $A \otimes B$ of A and B is a unital, generating subspace of $\mathcal{A} \otimes_\gamma \mathcal{B}$ and we may ask about the relation between $\text{bd}(A \otimes B)$ and the two sets $\text{bd}(A)$ and $\text{bd}(B)$.

The question is easy to answer in the commutative case. If $\mathcal{A} = C(X)$ and $\mathcal{B} = C(Y)$ then $\mathcal{A} \otimes_\gamma \mathcal{B} = C(X \times Y)$ and

$$\text{bd}(A \otimes B) = \text{bd}(A) \times \text{bd}(B).$$

In this paper we shall show that the same result holds anytime one of the factors is a GCR algebra. In the case in which \mathcal{B} is the algebra, M_n , of $n \times n$ matrices, this result is essentially contained in [3].

If \mathcal{A} and \mathcal{B} are C^* -algebras, if γ is any C^* -cross norm on $\mathcal{A} \otimes \mathcal{B}$, and if π_1 (resp. π_2) is an irreducible representation of \mathcal{A} (resp. \mathcal{B}), then $\pi_1 \otimes_\gamma \pi_2$ is an irreducible representation of $\mathcal{A} \otimes_\gamma \mathcal{B}$. We first show that if $\pi_1 \otimes_\gamma \pi_2$ is a boundary representation, then so are π_1 and π_2 .

LEMMA 1. *Suppose that A is a unital, generating subspace for \mathcal{A} , that B is a unital, generating subspace for \mathcal{B} and that $\pi_1 \otimes_\gamma \pi_2$ is a boundary representation for $A \otimes B$. Then $\pi_1 \in \text{bd}(A)$ and $\pi_2 \in \text{bd}(B)$.*

PROOF. Suppose the contrary. Then one of the factors, say π_1 , is not a boundary representation. Hence there exists a completely positive linear map $\phi_1: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_1)$ such that $\phi_1 \neq \pi_1$ but $\phi_1|_A = \pi_1|_A$. (Here, \mathcal{H}_1 is the Hilbert space on which π_1 acts.) By means of the Stinespring representation for completely positive maps [6], one can show that the tensor product of two completely positive maps is again completely positive. Therefore $\phi_1 \otimes_\gamma \pi_2$ is a completely positive extension of $\pi_1 \otimes_\gamma \pi_2|_{A \otimes B}$ which is unequal to $\pi_1 \otimes_\gamma \pi_2$. This contradicts the assumption that $\pi_1 \otimes_\gamma \pi_2 \in \text{bd}(A \otimes B)$.

In general, not every irreducible representation π on $\mathcal{A} \otimes_\gamma \mathcal{B}$ factors as a product $\pi_1 \otimes_\gamma \pi_2$ of irreducible representations; if, however, we assume that one of the C^* -algebras is a GCR algebra, then every irreducible representation does factor [2]. Note, also, that since GCR algebras are nuclear, there is a unique C^* -crossnorm on $\mathcal{A} \otimes \mathcal{B}$, which we denote by $\mathcal{A} \otimes_m \mathcal{B}$. Thus, when one of the factors is GCR, Lemma 1 asserts $\text{bd}(A \otimes B) \subseteq \text{bd}(A) \times \text{bd}(B)$, provided that we identify the pair (π_1, π_2) with the product $\pi_1 \otimes_m \pi_2$.

The following lemma is probably known, but no reference for it could be found.

LEMMA 2. *Let \mathcal{S} be a unital C^* -algebra contained in another C^* -algebra \mathcal{T} (with the same unit). Let ϕ be a completely positive map on \mathcal{T} , and let π be a representation of \mathcal{S} such that $\phi|_{\mathcal{S}} = \pi$. Then $\phi(ts) = \phi(t)\pi(s)$ and $\phi(st) = \pi(s)\phi(t)$, for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$.*

PROOF. Let $\phi = V^* \sigma V$ be the Stinespring representation for ϕ (see [6]), and let $P = VV^*$ be the range projection of V . Observe that $P\sigma P$ restricted to \mathcal{S} is unitarily equivalent to $\phi|_{\mathcal{S}} = \pi$. In particular, $P\sigma P$ is multiplicative on \mathcal{S} , so

that P is semi-invariant for $\sigma(\mathfrak{S})$, i.e. P is the difference between a pair of nested invariant projections for $\sigma(\mathfrak{S})$. (See [5].) Since $\sigma(\mathfrak{S})$ is a C^* -algebra, P is in fact reducing for $\sigma(\mathfrak{S})$; thus P commutes with $\sigma(\mathfrak{S})$. For any $s \in \mathfrak{S}$ and $t \in \mathfrak{T}$ we then have

$$\begin{aligned} \phi(st) &= V^*\sigma(st)V = V^*\sigma(s)\sigma(t)V \\ &= V^*P\sigma(s)\sigma(t)V = V^*\sigma(s)P\sigma(t)V \\ &= V^*\sigma(s)V V^*\sigma(t)V = \phi(s)\phi(t) = \pi(s)\phi(t). \end{aligned}$$

A similar equality proves $\phi(ts) = \phi(t)\pi(s)$.

As an immediate consequence of this lemma we have the following:

COROLLARY. *Let π_1 and π_2 be representations of C^* -algebras \mathfrak{A} and \mathfrak{B} , respectively. Let ϕ be a completely positive map defined on $\mathfrak{A} \otimes_{\gamma} \mathfrak{B}$ and assume that $\phi(1 \otimes b) = 1 \otimes \pi_2(b)$, for all $b \in \mathfrak{B}$. Then*

$$\phi(a \otimes b) = \phi(a \otimes 1)(1 \otimes \pi_2(b)) = (1 \otimes \pi_2(b))\phi(a \otimes 1),$$

for all $a, b \in \mathfrak{B}$. If, in addition, we assume that $\phi(a \otimes 1) = \pi_1(a) \otimes 1$, for all $a \in \mathfrak{A}$, then $\phi = \pi_1 \otimes_{\gamma} \pi_2$.

LEMMA 3. *Let A and B be unital, generating subspaces of C^* -algebras \mathfrak{A} and \mathfrak{B} , respectively. Let $\pi_1 \in \text{bd}(A)$ and $\pi_2 \in \text{bd}(B)$. Then $\pi_1 \otimes_{\gamma} \pi_2 \in \text{bd}(A \otimes B)$.*

PROOF. Let \mathcal{H}_1 and \mathcal{H}_2 be the Hilbert spaces on which π_1 and π_2 act. Then $\pi_1 \otimes_{\gamma} \pi_2$ acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let ϕ be a completely positive linear mapping of $\mathfrak{A} \otimes_{\gamma} \mathfrak{B}$ into $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\phi(x) = \pi_1 \otimes_{\gamma} \pi_2(x)$, for all $x \in A \otimes B$. We must prove that $\phi = \pi_1 \otimes_{\gamma} \pi_2$. By the corollary to Lemma 2, it suffices to prove that ϕ agrees with $\pi_1 \otimes_{\gamma} \pi_2$ on each of the subalgebras $1 \otimes \mathfrak{B}$ and $\mathfrak{A} \otimes 1$.

Let E be any rank one projection in $\mathcal{L}(\mathcal{H}_2)$. The mapping

$$a \rightarrow (1 \otimes E)\phi(a \otimes 1)(1 \otimes E)$$

is the composition of three completely positive mappings and hence is itself a completely positive mapping defined on \mathfrak{A} . Let e be a unit vector in the range of E and let \mathcal{K} be the range of $1 \otimes E$. Then the transformation U defined by $Ux = x \otimes e$, $x \in \mathcal{H}_1$, is a unitary mapping of \mathcal{H}_1 onto \mathcal{K} . Let $\hat{\pi}(a) = U\pi_1(a)U^*$, $a \in \mathfrak{A}$. Note that $\hat{\pi}(a)$ is the restriction to \mathcal{K} of

$$\pi_1(a) \otimes E = (1 \otimes E)(\pi_1(a) \otimes 1)(1 \otimes E).$$

Since $\hat{\pi}$ is unitarily equivalent to π_1 , $\hat{\pi} \in \text{bd}(A)$. Let $\psi(a)$ be the restriction to \mathcal{K} of $(1 \otimes E)\phi(a \otimes 1)(1 \otimes E)$. Then ψ is completely positive and agrees with $\hat{\pi}$ on A , hence on all of \mathfrak{A} .

Let $x, y \in \mathcal{H}_1$ and $r \in \mathcal{H}_2$. The paragraph above asserts that, for any $a \in \mathfrak{A}$,

$$\langle \phi(a \otimes 1)(x \otimes r), y \otimes r \rangle = \langle (\pi_1(a) \otimes 1)(x \otimes r), y \otimes r \rangle.$$

(Just let E be the rank one projection on the subspace spanned by r .) Let

$D = \phi(a \otimes 1) - \pi_1(a) \otimes 1$. So, we have $\langle D(x \otimes r), y \otimes r \rangle = 0$, for all $x, y \in \mathcal{H}_1, r \in \mathcal{H}_2$. The polarization formula

$$\begin{aligned} 4\langle D(x \otimes r), y \otimes s \rangle &= \langle D(x \otimes (r + s)), y \otimes (r + s) \rangle \\ &\quad - \langle D(x \otimes (r - s)), y \otimes (r - s) \rangle \\ &\quad + i\langle D(x \otimes (r + is)), y \otimes (r + is) \rangle \\ &\quad - i\langle D(x \otimes (r - is)), y \otimes (r - is) \rangle \end{aligned}$$

yields $\langle D(x \otimes r), y \otimes s \rangle = 0$, for all $x, y \in \mathcal{H}_1$ and all $r, s \in \mathcal{H}_2$. Consequently, if $z_1 = \sum_1^n x_i \otimes r_i$ and $z_2 = \sum_1^m y_i \otimes s_i$, then $\langle Dz_1, z_2 \rangle = 0$. Since z_1, z_2 run through a dense subset of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and D is bounded, we obtain $D = 0$. Thus $\phi(a \otimes 1) = \pi_1(a) \otimes 1$, for all $a \in \mathcal{A}$. The equality $\phi(1 \otimes b) = 1 \otimes \pi_2(b)$, for all $b \in \mathcal{B}$ is obtained in the same way. This proves the lemma.

If we keep in mind the fact that if either \mathcal{A} or \mathcal{B} is a GCR algebra then any irreducible representation of $\mathcal{A} \otimes_m \mathcal{B}$ is the tensor product of irreducible representations of \mathcal{A} and \mathcal{B} we obtain the following theorem.

THEOREM. *Let A and B be unital, generating subspaces of C^* -algebras \mathcal{A} and \mathcal{B} , respectively. Assume that either \mathcal{A} or \mathcal{B} is a GCR algebra. Then $\text{bd}(A \otimes B) = \text{bd}(A) \times \text{bd}(B)$.*

BIBLIOGRAPHY

1. W. Arveson, *Subalgebras of C^* -algebras*, Acta Math. **123** (1969), 141–224.
2. A. Guichardet, *Tensor products of C^* -algebras*, Aarhus University Lecture Notes Series, No. 12, Aarhus, 1969.
3. A. Hopenwasser, *Boundary representations on C^* -algebras with matrix units*, Trans. Amer. Math. Soc. **177** (1973), 483–490.
4. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966.
5. D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. (Szeged) **26** (1956), 289–299.
6. W. Stinespring, *Positive functions on C^* -algebras*, Proc. Amer. Math. Soc., **6** (1955), 211–216.

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