PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

A NOTE ON THE DISINTEGRATION OF MEASURES

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ABSTRACT. Complements to a theorem of Bourbaki on the disintegration of measures on separated topological spaces.

If X and T are separated topological spaces, ν is a measure on X, and p is a mapping of X into T, then, under suitable hypotheses, there exist a measure $\mu = p(\nu)$ on T and a family $(\lambda_t)_{t \in T}$ of measures on X indexed by T, such that

(a) for every $t \in T$, λ_t is carried by $p^{-1}(\{t\})$, and

(b) $\nu = \int \lambda_t \, d\mu(t)$ in an appropriate sense; moreover,

(c) $\|\lambda_t\| = 1$ for locally μ-almost every t in T, and

(d) the family $(\lambda_t)_{t \in T}$ is determined by properties (a) and (b) up to a locally μ-negligible set [**2**, §2, No. 7, Proposition 13].

The purpose of this note is to repair a gap in the proofs of (c) and (d) given in [**2**]. {To pinpoint the problem: in the proof of (c) on p. 42 of [**2**], the application of (12) is unjustified since the function $f\phi_A$ need not be universally measurable.} For greater clarity, we shall operate under somewhat weaker hypotheses than [**2**] (to which we refer for notations and terminology).

For the rest of the paper, X and T denote separated topological spaces, ν is a measure on X (all measures under discussion are positive), μ is a measure on T, and $t \mapsto \lambda_t$ is a mapping of T into the set of measures on X. If $f \in \mathcal{F}_+(X)$ (that is, f is a positive numerical function on X), we define positive numerical functions h_f and h_f^* on T by the formulas $h_f(t) = \lambda_t^{\bullet}(f)$, $h_f^*(t) = \lambda_t^*(f)$ (thus $0 \leqslant h_f \leqslant h_f^*$). We assume that the mapping $t \mapsto \lambda_t$ satisfies the following condition (obviously weaker than the condition (b) of [**2**, §2, No. 7, Proposition 13]):

(*) *If $f \in \mathcal{F}_+(X)$ is lower semicontinuous or upper semicontinuous, then h_f is μ-measurable and $\mu^{\bullet}(h_f) = \nu^{\bullet}(f)$.*

In more florid notation, the formula in (*) means that

$$\int_X^{\bullet} f(x)\, d\nu(x) = \int_T^{\bullet} d\mu(t) \int_X^{\bullet} f(x)\, d\lambda_t(x).$$

LEMMA 1. *For every $f \in \mathcal{F}_+(X)$, one has $\nu^*(f) \geqslant \mu^{\bullet}(h_f^*) \geqslant \mu^{\bullet}(h_f)$.*

PROOF. Formally the same as in [**1**, §3, No. 2, Proposition 3]. □

LEMMA 2. *If $f \in \mathcal{F}_+(X)$ is ν-negligible, then f is λ_t-negligible for locally μ-almost every t in T.*

Received by the editors July 15, 1977.

AMS (MOS) subject classifications (1970). Primary 28A50.

PROOF. Immediate from Lemma 1 (cf. [**1**, §3, No. 2, Corollary 1 of Proposition 3]). □

LEMMA 3. *If $f \in \mathcal{F}_+(X)$ is ν-moderated, then f is λ_t-moderated for locally μ-almost every t in T.*

PROOF. Write $f = \sum_{n \geqslant 1} f_n$, where f_1 is ν-negligible and, for each $n \geqslant 2$, f_n vanishes outside a compact set [**2**, §1, No. 9, Corollary 3 of Proposition 14]; the proof is then formally the same as [**1**, §3, No. 2, Corollary 2 of Proposition 3]. □

Assertions (ii) and (iii) of the following proposition fill the gap mentioned at the beginning of the paper (cf. [**2**, §2, No. 7, Remark 2]):

PROPOSITION. *Assuming condition* (*) *is verified, if ν is moderated and $f \in \mathcal{F}_+(X)$ is ν-measurable, then* (i) *f is λ_t-measurable and λ_t-moderated for locally μ-almost every t in T,* (ii) *the function h_f: $t \mapsto \lambda_t^{\bullet}(f)$ is μ-measurable, and* (iii) $\mu^{\bullet}(h_f) = \nu^{\bullet}(f)$.

PROOF. (cf. [**1**, §3, No. 2, Proposition 5]). Let $(X_n)_{n \geqslant 1}$ be a denumerable ν-crushing of X such that the restriction of f to each of the compact sets X_n is continuous [**2**, §1, No. 8, Propositions 10, 11].

(i) In view of Lemma 3, we need only show that f is λ_t-measurable for locally μ-almost every t. The proof is formally the same as in [**1**, §3, No. 2, Proposition 4a)].

(ii), (iii) Let $N = X - \bigcup X_n$; thus $\nu^*(N) = \nu^{\bullet}(N) = 0$. Write $f_0 = f\varphi_N$ and $f_n = f\varphi_{X_n}$ for $n \geqslant 1$. The conclusions (ii) and (iii) hold for f_0 by Lemma 2. For $n \geqslant 1$, f_n is upper semicontinuous, hence conditions (ii) and (iii) hold for f_n by the hypothesis (*). Let S be the set of $t \in T$ such that f_0 is not λ_t-negligible; by Lemma 2, $\mu^{\bullet}(S) = 0$. For $n \geqslant 1$, f_n is universally measurable. It follows that if $t \in T - S$, then f_n is λ_t-measurable for all $n \geqslant 0$, hence so is f, whence $h_f(t) = \sum_n h_{f_n}(t)$ [**2**, §1, No. 5, Proposition 4]. Thus $h_f = \sum_n h_{f_n}$ locally μ-almost everywhere, therefore h_f is also μ-measurable and $\mu^{\bullet}(h_f) = \sum_n \mu^{\bullet}(h_{f_n}) = \sum_n \nu^{\bullet}(f_n) = \nu^{\bullet}(f)$ by [**2**, §1, No. 5, Proposition 4]. □

COROLLARY. *Assuming condition* (*) *is verified, if ν is moderated and $f \geqslant 0$ is ν-integrable, then h_f is essentially μ-integrable, f is λ_t-integrable for locally μ-almost every t in T, and $\nu^{\bullet}(f) = \mu^{\bullet}(h_f)$, that is,*

$$\int f(x)\, d\nu(x) = \int d\mu(t) \int f(x)\, d\lambda_t(x).$$

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