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ENTIRE FUNCTIONS WITH ALMOST RADIALLY DISTRIBUTED VALUES

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ABSTRACT. Let $f(z)$ be an entire function of finite lower order. Assume that there exist a positive number h and an unbounded sequence $\{w_n\}_{n=1}^{\infty}$ such that all roots of the equations $f(z) = w_n$ $(n = 1, 2, \ldots)$ lie in $\{z; |\operatorname{Im} z| < h\}$. Then $f(z)$ is a polynomial of degree not greater than two. The hypothesis of the finiteness of lower order of $f(z)$ cannot be removed.

1. Edrei [**2**] studied meromorphic functions with three radially distributed values and he proved the following elegant theorem.

THEOREM A. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{w_n\}_{n=1}^{\infty}$ such that all the roots of the equations $f(z) = w_n$ $(n = 1, 2, \ldots)$ be real. Then $f(z)$ is a polynomial of degree not greater than two.*

We extend Theorem A and show the following.

THEOREM. *Let $f(z)$ be an entire function of finite lower order. Assume that there exist a positive number h and an unbounded sequence $\{w_n\}_{n=1}^{\infty}$ such that all roots of the equations $f(z) = w_n$ $(n = 1, 2, \ldots)$ lie in $\{z; |\operatorname{Im} z| < h\}$. Then $f(z)$ is a polynomial of degree not greater than two.*

We note that the hypothesis of the finiteness of the lower order of $f(z)$ cannot be removed from our Theorem. In fact, Fuchs and Hayman [**4**, p. 81] proved that there exists an entire function $f(z)$, such that in the strip $A = \{z = x + iy;\ x > 0,\ |y| \leqslant \pi\}$ $f(z) = \exp(e^z + z) + O(z^{-2})$, while outside $A, f(z) = O(z^{-2})$ uniformly as $z \to \infty$.

The proof of the Theorem goes in two stages: (1) $f(z)$ is an entire function of order at most one, (2) $f(z)$ is a polynomial. The proof of (1) follows closely the corresponding steps in the proof of Theorem 1 in [**1**]. The proof of (2) is quite different from that of Edrei.

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2. Lemmas. Our starting point will be the following three lemmas obtained by Edrei.

Consider the q radii defined by

$$re^{i\omega_1}, \ldots, re^{i\omega_q} \quad (r \geqslant 0)$$

where $0 \leqslant \omega_1 < \cdots < \omega_q < 2\pi$ $(q \geqslant 1)$.

LEMMA A [1]. *Let $\Delta_k(\delta)$ be the sector defined by*

$$r > 1, \quad \omega_k + \delta < \theta < \omega_{k+1} - \delta \qquad (k = 1, \ldots, q), \left[\omega_{q+1} = 2\pi + \omega_1\right]$$

and write $\pi/\gamma = \omega_{k+1} - \omega_k - 2\delta$. Consider the conformal transformation

$$w = \frac{u^\gamma - u^{-\gamma} - \tau}{u^\gamma - u^{-\gamma} + \tau} = \phi_k^{-1}(u)$$

where τ is a positive parameter and $u = e^{-i\zeta_k}z$, $\zeta_\kappa = (\omega_k + \omega_{k+1})/2$. Then the function $z = e^{i\zeta_k}\phi_k(w)$ maps the circle $|w| < 1$ onto the sector $\Delta_k(\delta)$, and we have

$$\tau/4r^{-\gamma}\cos(\gamma[\theta - \zeta_k]) < 1 - |w| < 8\tau r^{-\gamma} \quad (|z| = r).$$

The following Lemma B is a modification of Lemma 1 in [1].

LEMMA B. *Let $f(z)$ be meromorphic in the region $\{z;\ 1 < |z| < +\infty\}$ and assume that for any $\delta > 0$ all but a finite number of the roots of the three equations $f(z) = 0$, $f(z) = \infty$, $f^{(m)}(z) = 1$ $(m \geqslant 0, f^{(0)} = f)$ lie in the angles $|\arg z - \omega_k| < \delta$ $(k = 1, \ldots, q)$. Consider the q functions defined by $F_k(w) = f(e^{i\zeta_k}\phi_k(w))$ $(k = 1, \ldots, q)$. Then as $t \to 1$ $(0 < t < 1)$*

$$m(t, F_k(w)) = O\left(\log \frac{1}{1-t}\right), \quad m\left(t, F_k(w)^{-1}\right) = O\left(\log \frac{1}{1-t}\right),$$

$$m\left(t, \frac{1}{f^{(m)}\left(e^{i\zeta_k} \cdot \phi_k(w)\right) - 1}\right) = O\left(\log \frac{1}{1-t}\right).$$

LEMMA C [2]. *Let $G(t)$ be a positive, real, continuous and nondecreasing function defined for $t \geqslant t_0 > 0$. Assume that the order ρ and the lower order μ of $G(z)$ satisfy $\mu < \rho$ $(\rho \leqslant +\infty)$ and let σ, τ be given such that $\mu < \sigma < \tau < \rho$. Then there exist arbitrary large values of r such that*

$$\frac{G(r)}{r^\tau} \geqslant \frac{G(t)}{t^\tau} \quad \left(t_0 \leqslant t \leqslant r^{\tau/\sigma}\right), \quad \frac{G(r)}{r^\tau} \geqslant 1.$$

LEMMA D. *Let $h(z)$ ($\neq$ constant) be a meromorphic function of finite lower order μ, order ρ $(\leqslant +\infty)$, the poles of which have a positive deficiency. Then there exists a sequence of Pólya peaks $\{r_n\}$ of finite order τ of $T(r, h)$ such that $\mu \leqslant \tau \leqslant \rho$ and*

$$\tau \leqslant \liminf_{n\to\infty} \frac{\log T(r_n, h)}{\log r_n}, \tag{2.1}$$

and further there exist two positive numbers K and A such that

$$\text{measure } J(r_n) > A \tag{2.2}$$

for all sufficiently large n where

$$J(r_n) = \{\theta;\, 0 \leqslant \theta < 2\pi,\, |h(r_n e^{i\theta})| > \exp[K \cdot T(r_n, h)]\}.$$

PROOF. We can define a sequence of Pólya peaks $\{r_n\}$ of finite order τ ($\mu \leqslant \tau \leqslant \rho$) satisfying (2.1) in view of Lemma C by a routine argument [6]. Now we shall prove that the sequence satisfies (2.2). If (2.2) were false, then there would exist sequences $\varepsilon_n \to 0$, $\lambda_n \to 0$ such that for infinitely many n

$$\text{measure}\{\theta;\, 0 \leqslant \theta < 2\pi,\, |h(r_n e^{i\theta})| > \exp[\varepsilon_n T(r_n, h)]\} < \lambda_n.$$

Using a result of Edrei and Fuchs [3, p. 322], we obtain

$$m(r_n, h) \leqslant \varepsilon_n T(r_n, h) + 22T(2r_n, h)\lambda_n\{1 + \log^+ 1/\lambda_n\}.$$

Since $\{r_n\}$ is a sequence of Pólya peaks, we have $m(r_n, h) = o(T(r_n, h))$ $(n \to \infty)$ which conflicts with the fact that the poles of $h(z)$ have a positive deficiency. Thus we have proved the lemma.

Using the notations of Lemma A, we prove the following.

LEMMA E. *Let $h(z)$ be a meromorphic function of finite lower order μ and of order ρ (not necessarily finite). Assume that for any $\delta > 0$ all but a finite number of the poles of $h(z)$ lie in the angles $\{z;\, |\arg z - \omega_k| < \delta\}$ $(k = 1, \ldots, q)$ and that the poles of $h(z)$ have a positive deficiency. Then*

$$\rho > \beta = \sup\left\{ \frac{\pi}{\omega_2 - \omega_1}, \ldots, \frac{\pi}{\omega_{q+1} - \omega_q} \right\}$$

implies

$$\limsup_{t \to 1} \frac{m\big(t, h(e^{i\zeta_k}\phi_k(w))\big)}{\log 1/(1-t)} = +\infty \quad (0 < t < 1), \tag{2.3}$$

for some integer k $(1 \leqslant k \leqslant q)$.

PROOF. Choose δ sufficiently small such that $\delta < A/2q$ where A is the positive number defined in Lemma D, and such that

$$\rho \geqslant \rho' > \gamma' + \eta, \quad \rho' \geqslant \mu, \quad \rho' < +\infty \tag{2.4}$$

in view of the assumption $\beta < \rho$ where

$$\gamma' = \sup\left\{ \frac{\pi}{\omega_2 - \omega_1 - 2\delta}, \ldots, \frac{\pi}{\omega_{q+1} - \omega_q - 2\delta} \right\}$$

and η is a positive number. If we choose a sequence of Pólya peaks $\{r_n\}$ of order ρ' of $h(z)$ defined in Lemma D satisfying

$$\liminf_{n \to \infty} \frac{\log T(r_n, h)}{\log r_n} \geqslant \rho', \tag{2.5}$$

then in view of (2.2) we may associate with each r_n $(n \geqslant n_0)$ at least one argument θ_n belonging to one of the q arcs defined by $\omega_k + A/2q \leqslant \theta \leqslant \omega_{k+1} - A/2q$ and such that

$$\log|h(r_n e^{i\theta_n})| > K \cdot T(r_n, h). \tag{2.6}$$

We choose among the q sectors $\Delta_1(\delta), \ldots, \Delta_q(\delta)$ which are defined in Lemma A, a sector $\Delta_k(\delta)$ which contains an infinity of terms of $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}, \ldots$. Renumbering if necessary the elements of this sequence, we may assume that all its terms belong to $\Delta_k(\delta)$. Hence, using the notations of Lemma A, we obtain

$$\cos \gamma[\theta_n - \zeta_k] > \cos(\pi/2 - A\gamma/2q + \delta\gamma) \quad (> 0). \tag{2.7}$$

The function $H_k(w) = h(e^{i\zeta_k}\phi_k(w))$ is regular in the unit circle except for a finite number of points. If we put $w_n = \phi_k^{-1}(r_n e^{i(\theta_n - \zeta_k)})$, (2.6) becomes

$$\log|H_k(w_n)| > K \cdot T(r_n, h). \tag{2.8}$$

In view of (2.7) and the inequality of Lemma A we obtain

$$\frac{B}{r_n^\gamma} = \frac{\cos(\pi/2 - A\gamma/2q + \delta\gamma)}{4r_n^\gamma} < 1 - |w_n|. \tag{2.9}$$

Taking $t_n = |w_n| + (1 - |w_n|)/2$, we have

$$\begin{aligned} m(t_n, H_k(w)) &+ O\left(\log \frac{1}{1 - t_n}\right) \\ &\geqslant \frac{t_n - |w_n|}{t_n + |w_n|} \log|H_k(w_n)| \qquad (n \geqslant n_0), \end{aligned} \tag{2.10}$$

so that (2.8), (2.9) and (2.10) imply

$$m(t_n, H_k(w)) + O\left(\log \frac{1}{1 - t_n}\right) \geqslant \frac{BK}{4r_n^\gamma} T(r_n). \tag{2.11}$$

If the lemma were not true, then we would have

$$m(t, H_k(w)) = O\left(\log \frac{1}{1 - t}\right). \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$K' \log \frac{r_n^\gamma}{B} \geqslant \frac{BK}{4} \frac{T(r_n)}{r_n^\gamma}$$

where K' is a positive number, and hence $T(r_n) < r_n^{\gamma+\eta}$ provided n is large enough. Thus we have $\rho' \leqslant \gamma + \eta \leqslant \gamma' + \eta$ by (2.5) which contradicts (2.4). This contradiction proves the lemma.

Using Lemma B and Lemma E we can prove the following Lemma 1, by the reasonings similar to those of Theorem 1 in [**1**].

LEMMA 1. *Let $f(z)$ be a meromorphic function of finite lower order and such that for any $\delta > 0$ all but a finite number of roots of the three equations $f(z) = 0$, $f(z) = \infty$, $f^{(m)}(z) = 1$ lie in the angles $\{z; |\arg z - \omega_k| < \delta\}$ $(k = 1, \ldots, q)$. Denote by $\delta(a, f^{(m)})$ the deficiency of the value a of the function $f^{(m)}$ and assume*

$$\delta(0, f) + \delta(1, f^{(m)}) + \delta(\infty, f) > 0.$$

Then the order ρ of f is necessarily finite and $\rho \leqslant \beta$.

Further we quote the following lemma which is easily proved by a result of Heins [**5**, p. 70].

LEMMA F. *Suppose that $u(z)$ is a subharmonic function in the finite plane which is not negative and not identically constant. If the following conditions are fulfilled for $r > r_0$ ($\geqslant 0$): (1) $\inf_{|z|=r} u(z) = 0$, (2) the angular measure of $\{u(z) = 0,\ |z| = r\} \geqslant \phi_0$ ($\geqslant 0$). Then the order ρ_u of $u(z) = \limsup_{r\to\infty} \log \sigma(r)/\log r \geqslant \pi/(2\pi - \phi_0)$ where $\sigma(r) = \sup_{|z|=r} u(z)$.*

Using this lemma, we have

LEMMA 2. *Suppose that an unbounded region Ω is contained in an angular region $\Delta = \{z;\ |\arg z + \pi/2| < \alpha\}$. Let $h(z)$ be an arbitrary positive harmonic function with boundary value zero in Δ and let $u(z)$ be an arbitrary positive harmonic function with boundary value zero in Ω. Extending $h(z)$, $u(z)$ to the whole plane as the subharmonic functions by the standard method and denoting them by the same notations, we have $\rho_u \geqslant \rho_h$.*

PROOF. By the Picard principle $h(z)$ is represented as

$$h(z) = cr^{\pi/2\alpha} \cos(\theta + \pi/2)\pi/2\alpha, \quad z = re^{i\theta},$$

where c is a constant positive value. Hence $\rho_h = \pi/2\alpha$. Therefore ρ_u is estimated as follows by Lemma F,

$$\rho_u \geqslant \pi/(2\pi - \phi_0) \geqslant \pi/2\alpha = \rho_h.$$

3. Proof of the Theorem. The genus of $f(z)$ is not greater than one by Lemma 1. First we assume that the genus of $f(z)$ is one. Further we may assume that $w_1 = 0$ and $|w_1| < |w_2| < \cdots < |w_n| \to +\infty$ $(n \to \infty)$. Then $f(z)$ may be represented as

$$f(z) = \lambda e^{az} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j}\right)$$

where λ ($\neq 0$) is a constant. Since the genus of $f(z)$ is one, putting $z_j = r_j e^{i\theta_j}$ we have

$$\left|\sum_{j=1}^{\infty} \operatorname{Im} \frac{1}{z_j}\right| \leqslant \sum_{j=1}^{\infty} \frac{|r_j \sin \theta_j|}{r_j^2} < h \sum_{j=1}^{\infty} \frac{1}{r_j^2} < +\infty.$$

Therefore $f(z)$ may be rewritten as

$$f(z) = \lambda e^{bz} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) \exp\left\{\left(\operatorname{Re} \frac{1}{z_j}\right) z\right\}$$

where $b = a + i\sum_{j=1}^{\infty} \operatorname{Im} 1/z_j$. We may suppose $\operatorname{Im} b \geqslant 0$, otherwise we can replace $f(z)$ by $\overline{f(\bar{z})}$.

We consider

$$\frac{zf'(z)}{f(z)} = z\left\{\sum_{j=1}^{\infty} \left(\frac{1}{z - z_j} + \operatorname{Re} \frac{1}{z_j}\right) + b\right\}$$

in $\{z;\ |\arg z + \pi/2| < \alpha < \pi/2\}$. Let K be a positive number such that $K \cdot \sin\alpha \geqslant 2\pi$. Then there exists a positive number $r_1 = r_1(K)$ such that

$$\left|\frac{z \cdot f'(z)}{f(z)}\right| \geqslant |z| \left|\operatorname{Im} \frac{f'(z)}{f(z)}\right| > K \tag{3.1}$$

in $\{z;\ |\arg z + \pi/2| \leqslant \alpha,\ |z| > r_1\}$.

Next if the genus of $f(z)$ is zero, then (3.1) is similarly proved.

Now we can easily show that $|f(-iy)| \to +\infty$ as $y \to +\infty$. We choose w_n such that $|f(z)| < |w_n|$ for $|z| \leqslant r_1$. Let Ω be the component of $\{z;\ |f(z)| > |w_n|\}$ which contains the negative imaginary axis from some point on. If $\Omega \subset \{z;\ |\pi/2 + \arg z| < \alpha < \pi/2\}$, then putting $u(z) = \log|f(z)| - \log|w_n|$ the order of $f(z)$ is greater than one by Lemma 2 which is impossible. Therefore we may assume that $\Omega \cap \{z;\ |\pi/2 + \arg z| \leqslant \alpha\}$ contains an arc of a level curve γ of $f(z)$ which joins a point of the ray $\arg z = -\pi/2 - \alpha$ to a point of the ray $\arg z = -\pi/2$ and lies in $\{z;\ |z| > r_1\}$. If the equation of γ is given in parametric form $z = z(t)$ $(0 \leqslant t \leqslant 1)$, then we have

$$w = f(z(t)), \quad \frac{1}{w} \cdot \frac{dw}{dt} = \frac{zf'(z)}{f(z)} \cdot \frac{z'(t)}{z},$$

and if we put $w = |w_n| e^{i\phi(t)}$, then we have

$$\frac{dw}{dt} = iw\,\frac{d\phi}{dt}.$$

Hence $|d\phi/dt| = |zf'(z)/f(z)| \cdot |z'(t)/z|$ and by (3.1) we have

$$\int_0^1 \left|\frac{d\phi}{dt}\right| dt \geqslant \frac{K}{r_0} \int_0^1 |z'(t)|\, dt \geqslant \frac{K}{r_0}\,(r_0 \sin\alpha) = K \cdot \sin\alpha \geqslant 2\pi$$

where $r_0 = \max_{0 \leqslant t \leqslant 1} |z(t)|$. In view of $f'(z) \neq 0$ on γ, as z traverses γ in the fixed direction, w traverses the circle Γ; $|w| = |w_n|$ in the fixed direction and ϕ increases or decreases at least $K \cdot \sin\alpha$. Thus w traverses the whole of Γ and in particular $f(z) = w_n$ for some point $z \in \gamma$. But this contradicts the assumption of the Theorem. Hence if $f(z)$ satisfies the conditions of the Theorem, then $f(z)$ must be a polynomial. Then it is easy to show that the degree of $f(z)$ is at most two.

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