PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

HOMOMORPHISMS OF LATTICES OF CONTINUOUS FUNCTIONS

R. MENA AND B. ROTH

ABSTRACT. We prove that for Y a compact Hausdorff space, every lattice homomorphism from $C(Y)$ to $C(X)$ which takes each constant function on Y to the same function on X is linear.

Let $C(X)$ be the space of real-valued continuous functions on the topological space X. Pointwise operations make $C(X)$ into a ring, an algebra, and a lattice. If ϕ: $X \to Y$ is continuous, then ϕ induces a map $C(\phi)$: $C(Y) \to C(X)$ defined for all $f \in C(Y)$ by

$$C(\phi)(f) = f \circ \phi.$$

Clearly $C(\phi)$ is a ring, algebra, and lattice homomorphism which fixes each constant function, i.e., $C(\phi)(r_Y) = r_X$ for all $r \in \mathbf{R}$. It is well known that for Y a compact Hausdorff space, every ring homomorphism from $C(Y)$ to $C(X)$ taking 1_Y to 1_X arises in this way.

THEOREM. *Suppose Y is a compact Hausdorff space and Φ: $C(Y) \to C(X)$ is a lattice homomorphism such that $\Phi(r_Y) = r_X$ for all $r \in \mathbf{R}$. Then there exists a continuous map ϕ: $X \to Y$ such that $\Phi = C(\phi)$.*

We regard a prime ideal P as a nonempty, proper lattice ideal with $f \in P$ or $g \in P$ whenever $f \wedge g \in P$. Following Kaplansky [**2**], we say that a prime ideal P in $C(X)$ is *associated* with a point $x \in X$ if $g \in P$ whenever $f \in P$ and $g(x) < f(x)$. For X compact, every prime ideal in $C(X)$ is associated with some point of X and this point is unique if X is also Hausdorff [**2**, Lemma 3]. Moreover, it is easy to see that if $P \subset Q$ where P, Q are prime ideals in $C(X)$, X a compact Hausdorff space, then P and Q are associated with the same point.

For $x \in X$, let δ_x: $C(X) \to \mathbf{R}$ be the point evaluation map defined for $f \in C(X)$ by

$$\delta_x(f) = f(x).$$

LEMMA. *Suppose X is a compact Hausdorff space and Φ: $C(X) \to \mathbf{R}$. Then Φ is a lattice homomorphism satisfying $\Phi(r_X) = r$ for all $r \in \mathbf{R}$ if and only if $\Phi = \delta_x$ for some $x \in X$.*

PROOF. For $a \in \mathbf{R}$, let $P_a = \{r \in \mathbf{R}: r \leqslant a\}$. Since $\{P_a: a \in \mathbf{R}\}$ is a chain

Received by the editors July 21, 1977 and, in revised form, December 15, 1977.
AMS (MOS) subject classifications (1970). Primary 06A70, 46E05; Secondary 46A40.

of prime ideals in $\mathbf{R}$, $\{\Phi^{-1}(P_a)\colon a \in \mathbf{R}\}$ is a chain of prime ideals in $C(X)$ and hence each $\Phi^{-1}(P_a)$ is associated with the same point $x \in X$. We show that $\Phi(f) = f(x)$ for all $f \in C(X)$, i.e., $\Phi = \delta_x$. For $r \in \mathbf{R}$, $r < f(x)$ implies $r_X \in \Phi^{-1}(P_{\Phi(f)})$ since $\Phi^{-1}(P_{\Phi(f)})$ is associated with x and thus $r = \Phi(r_X) \leqslant \Phi(f)$. Hence $f(x) \leqslant \Phi(f)$. And for $r \in \mathbf{R}$, $f(x) < r$ implies $f \in \Phi^{-1}(P_r)$ so $\Phi(f) \leqslant r$. Therefore $\Phi(f) = f(x)$.

Proof of Theorem. For each $x \in X$, $\delta_x \circ \Phi\colon C(Y) \to \mathbf{R}$ is a lattice homomorphism fixing the constants and thus there exists a unique $y \in Y$ with $\delta_x \circ \Phi = \delta_y$. Letting $\phi(x) = y$, we have for all $f \in C(Y)$ and all $x \in X$

$$\Phi(f)(x) = (\delta_x \circ \Phi)(f) = \delta_y(f) = (f \circ \phi)(x)$$

and therefore $\Phi = C(\phi)$. The continuity of $\phi\colon X \to Y$ follows immediately since the zero sets $f^{-1}(0)$, $f \in C(Y)$, form a base for the closed sets of Y. Pertinent examples appear in [**1**].

References

1. K. Geba and Z. Semadeni, *On linear isotonical embedding of $C(\Omega_1)$ into $C(\Omega_2)$*, Studia Math. **19** (1960), 303–320.

2. I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Soc. **53** (1947), 617–623.

3. ______, *Lattices of continuous functions*. II, Amer. J. Math. **70** (1948), 626–634.

Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071