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EXTENSIONS OF A THEOREM OF FUGLEDE AND PUTNAM

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ABSTRACT. The operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal (theorem of Fuglede and Putnam). If X is of Hilbert-Schmidt class, the assumptions on A and B can be relaxed: it suffices that A and B^* be hyponormal, or that B be invertible with $\|A\|\,\|B^{-1}\| < 1$.

The classical Fuglede-Putnam theorem asserts that if A, B, X are operators in a Hilbert space such that $AX = XB$, and if A and B are normal, then also $A^*X = XB^*$ [**4**, Problem 152]. In this note we relax the hypotheses on A and B, at the cost of requiring X to be of Hilbert-Schmidt class. The resulting extensions of the Fuglede-Putnam theorem are perhaps unexciting, although it is somewhat surprising that normality can be dropped; of possibly greater interest is the curious method of proof. Our theorem is as follows:

THEOREM. *Suppose A, B, X are operators in the Hilbert space H, such that*

$$AX = XB. \tag{1}$$

Assume also that X is an operator of Hilbert-Schmidt class. Then

$$A^*X = XB^* \tag{2}$$

under either of the following hypotheses: (i) *A and B^* are hyponormal*; (ii) *B is invertible and* $\|A\|\,\|B^{-1}\| \leqslant 1$.

The proof employs what are essentially tensor product techniques. The operators in H of Hilbert-Schmidt class form an ideal $\mathcal{H}$ in the algebra $\mathcal{L}(H)$ of all operators in H, and $\mathcal{H}$ is itself a Hilbert space for the inner product

$$(X|Y) = \sum (Xe_\iota|Ye_\iota) = \mathrm{Tr}(Y^*X) = \mathrm{Tr}(XY^*),$$

where (e_ι) is any orthonormal basis of H [**3**, Chapter I, §6, n°6, Corollary of Theorem 5]. (One could identify $\mathcal{H}$ with $\overline{H} \otimes H$, where $\overline{H}$, awkwardly, is the conjugate Hilbert space of H; it seems simpler to work directly with $\mathcal{H}$.)

For each pair of operators $A, B \in \mathcal{L}(H)$, there is defined an operator $\mathcal{T} \in \mathcal{L}(\mathcal{H})$ via the formula $\mathcal{T}X = AXB$ (it is suggestive, though not strictly correct, to view $\mathcal{T}$ as the operator $A \otimes B$). Evidently $\|\mathcal{T}\| \leqslant \|A\|\,\|B\|$ (in fact equality holds, but we do not need it). The adjoint of $\mathcal{T}$ is given by the formula $\mathcal{T}^*X = A^*XB^*$, as one sees from the calculation $(\mathcal{T}^*X|Y) = (X|\mathcal{T}Y) = (X|AYB) = \mathrm{Tr}(XB^*Y^*A^*) = \mathrm{Tr}(A^*XB^*Y^*) = (A^*XB^*|Y)$. It follows at once that if A and B are both normal (resp. both selfadjoint), then

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so is $\mathfrak{T}$ (more precisely, see the lemma below). If $A \geqslant 0$ and $B \geqslant 0$, then also $\mathfrak{T} \geqslant 0$, as one sees from the calculation $(\mathfrak{T}X|X) = \mathrm{Tr}(AXBX^*) = \mathrm{Tr}(A^{1/2}XBX^*A^{1/2}) = \mathrm{Tr}[(A^{1/2}XB^{1/2})(A^{1/2}XB^{1/2})^*] \geqslant 0$; indeed, $\mathfrak{T}^{1/2}X = A^{1/2}XB^{1/2}$.

LEMMA. *If A and B^* are hyponormal, then the operator $\mathfrak{T}$ in $\mathcal{H}$ defined by $\mathfrak{T}X = AXB$ is also hyponormal.*

PROOF. We are assuming that $AA^* \leqslant A^*A$ and $B^*B \leqslant BB^*$, and we must show that $\mathfrak{T}\mathfrak{T}^* \leqslant \mathfrak{T}^*\mathfrak{T}$. Indeed, the formula

$$(\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*)X = A^*AXBB^* - AA^*XB^*B$$
$$= (A^*A - AA^*)XBB^* + AA^*X(BB^* - B^*B)$$

shows that $\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*$ is the sum of two positive operators. □

PROOF OF THE THEOREM. (i) Suppose first that, in addition to (i), B is invertible. Let $\mathfrak{T}$ be the operator in $\mathcal{H}$ defined by $\mathfrak{T}Y = AYB^{-1}$; since A and $(B^{-1})^* = (B^*)^{-1}$ are hyponormal [**2**, Chapter VI, §9, Exercise 11], it follows from the lemma that $\mathfrak{T}$ is hyponormal. Since $\mathfrak{T}X = X$ by (1), it follows that $\mathfrak{T}^*X = X$ [**2**, Chapter VII, §3, Exercise 5(i)], that is, $A^*X(B^{-1})^* = X$, whence (2). In the general case, choose a complex number λ such that $B - \lambda$ is invertible, and apply the preceding argument to $A - \lambda$, $B - \lambda$ in place of A, B.

(ii) The operator $\mathfrak{T}$ in $\mathcal{H}$ defined by $\mathfrak{T}Y = AYB^{-1}$ satisfies $\|\mathfrak{T}\| \leqslant 1$ by (ii), and $\mathfrak{T}X = X$ by (1), therefore $\mathfrak{T}^*X = X$ [**5**, §143], whence (2). (Incidentally, instead of (ii) one could suppose that for some λ, $B - \lambda$ is invertible and $\|A - \lambda\| \|(B - \lambda)^{-1}\| \leqslant 1$; or that A is invertible and $\|A^{-1}\| \|B\| \leqslant 1$.) □

If H is finite-dimensional, then every operator is of Hilbert-Schmidt class and every hyponormal operator is normal [**2**, Chapter VII, §2, Exercise 6], thus the assertion of case (i) of the theorem is the Fuglede-Putnam theorem for the finite-dimensional case; there is an even simpler trace argument for this case due to I. Kaplansky (cf. [**1**, Theorem 4]).

ADDENDUM. Part (i) of the theorem is related to a question raised by J. G. Stampfli and B. L. Wadhwa [*An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25** (1976), 359–365, Remark following Theorem 3]. The idea of (ii) is like that of Theorem 5.1 of the paper of C. Davis and W. M. Kahan [*The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal. **7** (1970), 1–46; MR **41** #9044]. I am indebted to the referee for these remarks.

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