

EXTENSIONS OF A THEOREM OF FUGLEDE AND PUTNAM

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ABSTRACT. The operator equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal (theorem of Fuglede and Putnam). If X is of Hilbert-Schmidt class, the assumptions on A and B can be relaxed: it suffices that A and B^* be hyponormal, or that B be invertible with $\|A\| \|B^{-1}\| < 1$.

The classical Fuglede-Putnam theorem asserts that if A, B, X are operators in a Hilbert space such that $AX = XB$, and if A and B are normal, then also $A^*X = XB^*$ [4, Problem 152]. In this note we relax the hypotheses on A and B , at the cost of requiring X to be of Hilbert-Schmidt class. The resulting extensions of the Fuglede-Putnam theorem are perhaps unexciting, although it is somewhat surprising that normality can be dropped; of possibly greater interest is the curious method of proof. Our theorem is as follows:

THEOREM. *Suppose A, B, X are operators in the Hilbert space H , such that*

$$AX = XB. \tag{1}$$

Assume also that X is an operator of Hilbert-Schmidt class. Then

$$A^*X = XB^* \tag{2}$$

under either of the following hypotheses: (i) A and B^ are hyponormal; (ii) B is invertible and $\|A\| \|B^{-1}\| \leq 1$.*

The proof employs what are essentially tensor product techniques. The operators in H of Hilbert-Schmidt class form an ideal \mathcal{H} in the algebra $\mathcal{L}(H)$ of all operators in H , and \mathcal{H} is itself a Hilbert space for the inner product

$$(X|Y) = \sum (Xe_i|Ye_i) = \text{Tr}(Y^*X) = \text{Tr}(XY^*),$$

where (e_i) is any orthonormal basis of H [3, Chapter I, §6, n^o6, Corollary of Theorem 5]. (One could identify \mathcal{H} with $\overline{H} \otimes H$, where \overline{H} , awkwardly, is the conjugate Hilbert space of H ; it seems simpler to work directly with \mathcal{H} .)

For each pair of operators $A, B \in \mathcal{L}(H)$, there is defined an operator $\mathfrak{T} \in \mathcal{L}(\mathcal{H})$ via the formula $\mathfrak{T}X = AXB$ (it is suggestive, though not strictly correct, to view \mathfrak{T} as the operator $A \otimes B$). Evidently $\|\mathfrak{T}\| \leq \|A\| \|B\|$ (in fact equality holds, but we do not need it). The adjoint of \mathfrak{T} is given by the formula $\mathfrak{T}^*X = A^*XB^*$, as one sees from the calculation $(\mathfrak{T}^*X|Y) = (X|\mathfrak{T}Y) = (X|AYB) = \text{Tr}(XB^*Y^*A^*) = \text{Tr}(A^*XB^*Y^*) = (A^*XB^*|Y)$. It follows at once that if A and B are both normal (resp. both selfadjoint), then

Received by the editors June 2, 1977.

AMS (MOS) subject classifications (1970). Primary 47A50; Secondary 47B20.

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so is \mathfrak{T} (more precisely, see the lemma below). If $A \geq 0$ and $B \geq 0$, then also $\mathfrak{T} \geq 0$, as one sees from the calculation $(\mathfrak{T}X|X) = \text{Tr}(AXBX^*) = \text{Tr}(A^{1/2}XB^{1/2}) = \text{Tr}[(A^{1/2}XB^{1/2})(A^{1/2}XB^{1/2})^*] \geq 0$; indeed, $\mathfrak{T}^{1/2}X = A^{1/2}XB^{1/2}$.

LEMMA. If A and B^* are hyponormal, then the operator \mathfrak{T} in \mathfrak{H} defined by $\mathfrak{T}X = AXB$ is also hyponormal.

PROOF. We are assuming that $AA^* \leq A^*A$ and $B^*B \leq BB^*$, and we must show that $\mathfrak{T}\mathfrak{T}^* \leq \mathfrak{T}^*\mathfrak{T}$. Indeed, the formula

$$\begin{aligned} (\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*)X &= A^*AXB^* - AA^*XB^*B \\ &= (A^*A - AA^*)XBB^* + AA^*X(BB^* - B^*B) \end{aligned}$$

shows that $\mathfrak{T}^*\mathfrak{T} - \mathfrak{T}\mathfrak{T}^*$ is the sum of two positive operators. \square

PROOF OF THE THEOREM. (i) Suppose first that, in addition to (i), B is invertible. Let \mathfrak{T} be the operator in \mathfrak{H} defined by $\mathfrak{T}Y = AYB^{-1}$; since A and $(B^{-1})^* = (B^*)^{-1}$ are hyponormal [2, Chapter VI, §9, Exercise 11], it follows from the lemma that \mathfrak{T} is hyponormal. Since $\mathfrak{T}X = X$ by (1), it follows that $\mathfrak{T}^*X = X$ [2, Chapter VII, §3, Exercise 5(i)], that is, $A^*X(B^{-1})^* = X$, whence (2). In the general case, choose a complex number λ such that $B - \lambda$ is invertible, and apply the preceding argument to $A - \lambda$, $B - \lambda$ in place of A , B .

(ii) The operator \mathfrak{T} in \mathfrak{H} defined by $\mathfrak{T}Y = AYB^{-1}$ satisfies $\|\mathfrak{T}\| \leq 1$ by (ii), and $\mathfrak{T}X = X$ by (1), therefore $\mathfrak{T}^*X = X$ [5, §143], whence (2). (Incidentally, instead of (ii) one could suppose that for some λ , $B - \lambda$ is invertible and $\|A - \lambda\| \|(B - \lambda)^{-1}\| \leq 1$; or that A is invertible and $\|A^{-1}\| \|B\| \leq 1$.) \square

If H is finite-dimensional, then every operator is of Hilbert-Schmidt class and every hyponormal operator is normal [2, Chapter VII, §2, Exercise 6], thus the assertion of case (i) of the theorem is the Fuglede-Putnam theorem for the finite-dimensional case; there is an even simpler trace argument for this case due to I. Kaplansky (cf. [1, Theorem 4]).

ADDENDUM. Part (i) of the theorem is related to a question raised by J. G. Stampfli and B. L. Wadhwa [*An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana Univ. Math. J. **25** (1976), 359–365, Remark following Theorem 3]. The idea of (ii) is like that of Theorem 5.1 of the paper of C. Davis and W. M. Kahan [*The rotation of eigenvectors by a perturbation*. III, SIAM J. Numer. Anal. **7** (1970), 1–46; MR **41** #9044]. I am indebted to the referee for these remarks.

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