

THE CANONICAL FORM OF A SCALAR OPERATOR ON A BANACH SPACE

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ABSTRACT. Let $A = \int \lambda dE(\lambda)$ be a scalar operator on a Banach space X . If there exists a vector $g \in X$ such that the closed convex hull of the range of the vector measure $\mu(\cdot) = E(\cdot)g$ has nonvoid interior, then A is similar to the operator $Qf(\lambda) = \lambda f(\lambda)$ on a quotient space of a suitably constructed \mathcal{L}^∞ space.

I. Introduction. Let A be a bounded, selfadjoint operator on the Hilbert space H with spectral representation $A = \int \lambda dE(\lambda)$. If A has simple spectra (i.e. there is some $g \in H$ so that the linear span of the range of the vector measure $\mu(\cdot) = E(\cdot)g$ is dense in H), then A is unitarily equivalent to multiplication by the independent variable on $\mathcal{L}^2(\sigma(A), \nu)$, where $\nu(M) = \langle \mu(M), g \rangle$ [1, p. 52]. The condition that A have simple spectra is in some sense a requirement that the range of the vector measure $E(\cdot)g$ be well dispersed in H . In order to motivate an alternative condition on the dispersion of the range of a vector measure we consider the following example. Let Q be the operation of multiplication by the independent variable on $\mathcal{L}^\infty([0, 1], \mu)$, i.e. $(Qf)(\lambda) = \lambda f(\lambda)$. The operator Q is a prespectral operator with spectral family $E(M)g = 1_M \cdot g$. Let $\mu(M) = E(M)1$, then the range of μ , $\mathcal{R}(\mu)$, is the set of characteristic functions of measurable subsets of $[0, 1]$. The positive portion of the unit ball of $\mathcal{L}^\infty([0, 1], \mu)$ is weak-* compact and the range of μ coincides with its extreme points. Therefore, the closed, convex hull of $\mathcal{R}(\mu)$ is all of the positive part of the ball; and thus, has nonvoid interior. We will say that a vector measure μ , taking values in a Banach space, is *full* provided $\overline{\text{co}}(\mathcal{R}(\mu))^0 \neq \emptyset$.

In this paper we shall show that, for a scalar operator $A = \int \lambda dE(\lambda)$ defined on a Banach space X , the existence of a vector $g \in X$ such that $\mu(\cdot) = E(\cdot)g$ is full implies that A is similar to $Qf(\lambda) = \lambda f(\lambda)$ on a quotient space of a suitably constructed \mathcal{L}^∞ space.

II.

DEFINITION 1. Let μ be a vector valued measure defined on a σ -algebra \mathfrak{B} and taking values in a Banach space X with norm $|\cdot|$. The *semivariation* of μ , denoted by $\|\mu\|$, is the scalar valued function on \mathfrak{B} defined by

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$$\|\mu\|(E) = \sup \left| \sum_{i=0}^n \alpha_i \mu(E_i) \right|$$

where the supremum is taken over all finite collections of scalars with $|\alpha_i| < 1$ and all finite partitions of E into disjoint members of \mathfrak{B} .

The support of μ will be denoted by $\text{supp}(\mu)$.

LEMMA 1 [2, p. 320]. *Let μ be a vector valued measure defined on a σ -algebra \mathfrak{B} . Then*

- (1) $\|\mu\|(E) \geq |\mu(E)| \geq 0$,
- (2) $\|\mu\|(E) < 4 \sup_{F \subseteq E} |\mu(F)| < \infty$,
- (3) $\|\mu\|(E) \leq \|\mu\|(F)$ if $E \subseteq F$,
- (4) *there is a positive scalar measure λ defined on \mathfrak{B} so that*
 - (a) $\lambda(E) \leq \|\mu\|(E)$,
 - (b) $\lim_{\lambda(E) \rightarrow 0} \|\mu\|(E) = 0$.

In all that follows λ will be a scalar measure guaranteed by (4) above. By (4) (b) of the lemma, $\lambda(E) = 0$ if and only if $\|\mu\|(E) = 0$, so for $F \in \mathfrak{B}$, $\mu - \text{ess sup}_{s \in F} |f(s)| = \lambda - \text{ess sup}_{s \in F} |f(s)|$; thus we denote by $\mathcal{L}^\infty(\mu)$, the space $\mathcal{L}^\infty(\lambda)$. The reason for this apparently artificial device is the desire to associate the space of functions with the measure μ itself.

DEFINITION 2. Let μ be a vector valued measure defined on a σ -algebra \mathfrak{B} of subsets of a set T . We denote by $\mathcal{L}^1(T, \mu)$ the set of all measurable complex valued functions f defined on T for which $\int f d\mu$ exists. For $f \in \mathcal{L}^1(T, \mu)$, $\mu_f(E) = \int_E f d\mu$ defines a vector measure on \mathfrak{B} . The norm on $\mathcal{L}^1(T, \mu)$ is then defined by

$$\|f\|_1 = \|\mu_f\|(T).$$

In the case that μ is scalar valued this space of functions is equivalent to the usual \mathcal{L}^1 space.

By [3, p. 45] $\mathcal{L}^1(T, \mu)$ is complete. In addition since $\lambda(E) \leq \|\mu\|(E) < \infty$ and

$$\left| \int_E f d\mu \right| \leq \left\{ \lambda - \text{ess sup}_{s \in E} |f(s)| \right\} \|\mu\|(E), \quad (1)$$

$$\mathcal{L}^\infty(\mu) \subset \mathcal{L}^1(T, \mu).$$

Let $C(T)$ denote the continuous complex valued functions on a subset T of the complex plane. We then have

THEOREM 1. $C(T)$ is dense in $\mathcal{L}^1(T, \mu)$.

PROOF. Let $A \in \mathfrak{B}$. By Lusin's theorem there exists a sequence of continuous functions g_n , with $|g_n(s)| \leq 1$, such that for $A_n = \{s \in T | g_n(s) \neq 1_A(s)\}$, $\lim_{n \rightarrow \infty} \lambda(A_n) = 0$. It follows that for $\epsilon > 0$ there exists an integer N so that for $n \geq N$, $\|\mu\|(A_n) \leq \epsilon/8$. For this N and $n \geq N$ we have

$$\begin{aligned} \|1_A - g_n\|_1 &\leq 4 \sup_{F \subseteq T} \left| \int_F (1_A - g_n) d\mu \right| = 4 \sup_{F \subseteq T} \left| \int_{F \cap A_n} (1 - g_n) d\mu \right| \\ &\leq 8 \sup_{F \subseteq T} \|\mu\|(F \cap A_n) \leq \epsilon. \end{aligned}$$

Thus, since simple functions are dense in $\mathcal{L}^1(T, \mu)$, the theorem holds. \square

We note that since

$$\|f\|_1 \leq 4 \sup_{F \subseteq T} \left| \int_F f d\mu \right| \leq 4 \left\{ \lambda - \text{ess sup}_{s \in T} |f(s)| \right\} \|\mu\|(T),$$

it follows that the polynomials together with their conjugates are dense in $\mathcal{L}^1(T, \mu)$ whenever T is a compact subset of the complex plane.

LEMMA 2 [3, p. 76]. *If μ is a vector valued measure defined on a σ -algebra \mathfrak{B} , taking values in a Banach space X , then*

$$\overline{\text{co}} \mathfrak{R}(\mu) = \left\{ \int f d\mu \mid f \text{ is measurable and } f(s) \in [0, 1] \text{ for all } s \in T \right\}.$$

LEMMA 3. *Let μ be a vector measure defined on a σ -algebra \mathfrak{B} of subsets of a set T , taking values in a Banach space X with norm $|\cdot|$. Then for every complex valued function defined on T , with $|f(s)| \leq 1$ for all $s \in T$, we have*

$$\left| \int_E f(s) d\mu(s) \right| \leq \|\mu\|(E).$$

The proof is standard, using measurable disjoint partitions of E .

THEOREM 2. *The linear operator $Q: \mathcal{L}^1(T, \mu) \rightarrow \mathcal{L}^1(T, \mu)$ defined by $Qf(s) = sf(s)$ is continuous if $\text{supp}(\mu)$ is contained in a bounded subset of the complex plane.*

PROOF. Clearly for $f \in \mathcal{L}^1$, $Qf \in \mathcal{L}^1$ so that the operator has domain all of \mathcal{L}^1 . Now suppose $\text{supp}(\mu) \subseteq \{z \in \mathbb{C} \mid |z| \leq r\} = D(r)$. Then for $E \subseteq D(r)$,

$$\left| \int_E sf(s) d\mu(s) \right| = \left| \int_E s d\mu_r(s) \right| = r \left| \int_E \frac{s}{r} d\mu_r(s) \right|.$$

Now $f(s) = s/r$ satisfies $|f(s)| \leq 1$, so by Lemma 3

$$\left| \int_E sf(s) d\mu(s) \right| \leq r \|\mu_r\|(X) = r \|f\|_1;$$

thus $\|Qf\|_1 = \|\mu_{Qf}\|(X) \leq r \|f\|_1$. \square

THEOREM 3. *Let $A = \int \lambda d\dot{E}(\lambda)$ be a bounded scalar operator defined on a Banach space X with $\sigma(A) = T$. For $g \in X$, let $\mu(\cdot) = E(\cdot)g$. Define $U: \mathcal{L}^1(T, \mu) \rightarrow X$ by $Uf = \int f d\mu$ and $Q: \mathcal{L}^1(T, \mu) \rightarrow \mathcal{L}^1(T, \mu)$ by $Qf(s) = sf(s)$. Then U is continuous and the following diagram commutes:*

$$\begin{array}{ccc} X & \xleftarrow{U} & \mathcal{L}^1(T, \mu) \\ A \downarrow & & \downarrow Q \\ X & \xleftarrow{U} & \mathcal{L}^1(T, \mu) \end{array} \tag{2}$$

PROOF. For any polynomial $p \in \mathcal{L}^1(T, \mu)$ we have $Up = p(A)$ so that

$$AUp = Ap(A) = UQp.$$

Since the polynomials are dense in $\mathcal{L}^1(T, \mu)$, the diagram commutes provided U and Q are continuous.

Since A is bounded, $\text{supp}(\mu) \subset \sigma(A)$ is bounded. Thus Q is continuous by Theorem 2. That U is continuous follows from

$$|Uf| = \left| \int f d\mu \right| \leq \sup_{E \subseteq T} \left| \int_E f d\mu \right| = \sup_{E \subseteq T} |\mu_f(E)| \leq \|\mu_f\|(T) = \|f\|_1. \quad \square$$

THEOREM 4. Let $A = \int \lambda dE(\lambda)$ be a scalar operator on a Banach space X with $T = \sigma(A)$. If there exists a vector $g \in X$ so that $\mu(\cdot) = E(\cdot)g$ is full, then A is similar to the multiplication operator on a factor space of $\mathcal{L}^\infty(T, \mu)$.

PROOF. Since $\mathcal{L}^\infty(T, \mu) \subset \mathcal{L}^1(T, \mu)$ the diagram (2) above commutes with $\mathcal{L}^\infty(T, \mu)$ replacing $\mathcal{L}^1(T, \mu)$. By (1) we see that U is continuous with the topology of $\mathcal{L}^\infty(T, \mu)$. Let N be the null-space of U . Since $AU = UQ$, then $Uf = 0$ implies $U(Qf) = 0$; thus $\hat{Q}: \mathcal{L}^\infty(T, \mu)/N \rightarrow \mathcal{L}^\infty(T, \mu)/N$ defined by $\hat{Q}[f] = [Qf]$ is well defined. Since μ is full and

$$\{f \mid f \text{ is measurable and } f(s) \in [0, 1] \text{ for all } s \in T\} \subseteq \mathcal{L}^\infty(T, \mu),$$

the range of U is all of X , and $\hat{U}: \mathcal{L}^\infty(T, \mu)/N \rightarrow X$, defined by $\hat{U}[f] = Uf$, is bijective with a continuous inverse (ensured by the Open Mapping Theorem). By the commutativity of (2), it follows that $A = \hat{U}\hat{Q}\hat{U}^{-1}$. \square

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