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APPROXIMATE UNITS IN IDEALS OF GROUP ALGEBRAS

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ABSTRACT. Several properties of locally compact groups can be characterized by the existence of certain approximate units in 3 ideals of the group algebra. We combine methods and results from [8], [7], [5].

Let G be a locally compact group, $M(G)$ the Banach algebra of all bounded complex measures, $L^1(G)$ the ideal consisting of all absolute-continuous measures with respect to left Haar-measure,

$$M^0(G) = \left\{ m \in M(G), \int 1\, dm = 0 \right\}, \quad L^0(G) = M^0(G) \cap L^1(G).$$

Let B be a Banach algebra. We consider the following properties: (U) B has a unit, (A) there exists a bounded net u_α in B such that $\lim_\alpha u_\alpha * f = f$ for every $f \in B$, (A_c) in addition $u_\alpha \in Z(B)$ the center of B, (A_d) $(u_\alpha * f - f) \in Z(B)$ for every $f \in B$.

We consider the following classes of locally compact groups: [SIN]: G has a basis of neighbourhoods of 1 invariant under all inner automorphisms, $[FC]^-$: every conjugacy class in G is precompact, $[FIA]^-$: the group of inner automorphisms is precompact in the group of all continuous automorphisms, $[FD]^-$, resp. FD: the group generated by all commutators $(xyx^{-1}y^{-1})$ is precompact, resp. finite (see [3]). G is amenable if there exists a left invariant mean on $L^\infty(G)$, the space of all measurable essentially bounded functions (see [2] or [8]).

It is well known that $L^1(G)$ always satisfies (A), and (U) iff G is discrete. Mosak, [7], proved that $L^1(G)$ satisfies (A_c) iff $G \in$ [SIN]. Reiter, [9], proved that $L^0(G)$ satisfies (A) iff G is amenable and Kotzmann, Rindler, [5], that $L^0(G)$ satisfies (A_c) iff $G \in [FIA]^-$.

Using their methods and results and well-known structure theorems we easily can complete the results above to the following list of characterizations:

Let G be a locally compact group then

1.	G is discrete	iff	$L^1(G)$ satisfies U
2.	G is compact	iff	$M^0(G)$ satisfies U
3.	G is finite	iff	$L^0(G)$ satisfies U

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4.	G is arbitrary l.c.	then	$L^1(G)$ satisfies A
5.	G is amenable	iff	$M^0(G)$ satisfies A
6.	G is amenable	iff	$L^0(G)$ satisfies A
7.	$G \in [\mathrm{SIN}]$	iff	$L^1(G)$ satisfies A_c
8.	$G \in [\mathrm{FC}]^-$	iff	$M^0(G)$ satisfies A_c
9.	$G \in [\mathrm{FIA}]^-$	iff	$L^0(G)$ satisfies A_c
10.	G is Abelian or discrete	iff	$L^1(G)$ satisfies A_d
11.	$G \in [\mathrm{FD}]^-$	iff	$M^0(G)$ satisfies A_d
12.	G is Abelian or discrete and $\in [\mathrm{FD}]$	iff	$L^0(G)$ satisfies A_d.

REMARKS. If the boundedness assumption of the net u_α is dropped then the following remarks apply: If we only assume that for every $f \in B$ and $\varepsilon > 0$ there exists $u = u(f, \varepsilon) \in B$ $\|u * b - u\| < \varepsilon$, then parts 5 and 6 remain open. The case of compact extensions of connected groups was settled in [**10**, Theorem 1 and Proposition 1]. A counterexample for discrete groups would lead to a nonamenable group not containing the free group of 2 generators, B. E. Johnson, [**4**]. If $u = u(f, \varepsilon)$ can be taken bounded independent of f and ε then (A) holds (see Altman [**1**] or Wichmann [**12**] who also proved that the same bound can be chosen for the net u_α). For $M^0(G)$ and $L^0(G)$ the best possible bound is 2.

PROOFS. If δ is the unit of $M(G)$, λ the normalized Haar measure if G is compact (resp. finite) then $\delta - \lambda$ is a unit of $M^0(G)$ (resp. $L^0(G)$). The proof of the converse in 3 contains proofs of 1 and 2: If $u \in L^0(G)$ satisfies $f * u = f$ for every $f \in L^0(G)$ we obtain for $m = \delta - u$, $\int 1\, dm = 1$ and $f * m = 0$, $f \in L^0(G)$. Choosing $f(z) = g(xz) - g(z)$, g continuous with compact support, $x \in G$ arbitrary, it follows from $0 = f * m(e) = \int f(y)\, dm(y^{-1})$ that m is a right Haar measure. $\int 1\, dm = 1$ implies G is compact, m is absolutely continuous and so is $\delta = u + m$ and G is also discrete. If u is a unit for $M^0(G)$ then in the same way we obtain that G is compact. If u is a unit for $L^1(G)$ then $0 = f * (\delta - u)(e)$, for every f continuous with compact support implies that $u = \delta \in L^1(G)$ and we obtain that G is discrete. The converse is clear.

The proof of 6 was given in [**9**], 5 follows analogously. The proof of 7 is contained in [**7**], that of 9 in [**5**].

PROOF OF 8. If $\lim \|u_\alpha * m - m\| = 0$ for every $m \in M^0(G)$, u_α central, then $v_\alpha = \delta - u_\alpha$ is central and $v_\alpha * m \to 0$ for every $m \in M^0(G)$. If w_α denotes the absolutely continuous part of v_α, then w_α also belongs to the center of $M(G)$ and putting $m = g - \delta$, $g \in L^1(G)$, $\int g = 1$, we obtain $\|w_\alpha - v_\alpha\| \leqslant \|v_\alpha * (g - \delta)\| \to 0$ $(v_\alpha * g \in L^1(G))$. This implies $\lim_\alpha \|w_\alpha * m\| = 0$ for every $m \in M^0(G)$, therefore $G \in [\mathrm{FC}]^-$ by [**5**, Proposition, (c)].

If conversely $G \in [\mathrm{FC}]^-$ then by Robertson's structure theorem ([**11**], for a proof see [**6**]) there exists a compact normal subgroup K such that $G/K = V \times D$, V a vector group, D a discrete [FC]-group. This reduces the problem to discrete groups, but then $M^0(G) = L^0(G)$ and the result follows from 9.

To prove the nontrivial part of 10 we assume that $(\delta - u_\alpha) * f$ is central for every $f \in L^1(G)$. Choosing f_β the normalized characteristic functions of U_β, a basis of neighbourhoods of $x \in G$, we obtain that $(\delta - u_\alpha) * \delta_x$ is central for every $x \in G$ (the center being weakly closed). The discrete part of this measure is also central and equal to δ_x if G is not discrete and G is necessarily Abelian.

If $G \in [\mathrm{FD}]^-$ then there exists a compact normal subgroup K such that G/K is Abelian. We obtain a suitable approximate unit for $M^0(G)$ by lifting from $M^0(G/K)$. Conversely we have $(\delta - u_\alpha) * m$ is central for every $m \in M^0(G)$. For a fixed α_0 we put $\mu = \delta - u_{\alpha_0}$. We want to show that $\mu * m$ is central for every $m \in M(G)$. We may assume that $\int 1\, dm = 1$, then $(m - u_\alpha) \in M^0(G)$ and $\mu * (m - u_\alpha)$ is central for every α. Furthermore $\lim_\alpha \mu * u_\alpha = \lim_\alpha u_\alpha * \mu = 0$. The center being closed this implies that $\mu * m$ and in particular $\mu * \delta_x$ is central for every $x \in G$. This implies that $\mu = \mu * \delta_{xyx^{-1}y^{-1}}$ and $\mu = \mu * \delta_z$ for every $z \in K$ the closure of the commutator subgroup. μ being a bounded measure, this implies that G is compact.

Proof of 12. As before, we obtain that $(\delta - u_\alpha) * \delta_x$ is central for all $x \in G$. By 10 we may assume that G is discrete and by 11 G is a [FC]-group. The converse is clear.

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