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ON THE ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES

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ABSTRACT. Let $f \in L[-\pi, \pi]$ be 2π-periodic. Noble **[6]** posed the following problem: if the fulfillment of some property of a function f on the whole interval $[-\pi, \pi]$ implies certain conclusions concerning the Fourier series $\sigma(f)$ of f, then what lacunae in $\sigma(f)$ guarantees the same conclusions when the property is fulfilled only locally? Applying the more powerful methods of approach to this kind of problems, originally developed by Paley and Wiener **[7]**, the absolute convergence of a certain lacunary Fourier series is studied when the function f satisfies some hypothesis in terms of either the modulus of continuity or the modulus of smoothness of order l considered only at a fixed point of $[-\pi, \pi]$. The results obtained here are a kind of generalization of the results due to Patadia **[8]**.

1. Introduction. Let

$$\sum_{k=1}^{\infty} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right) \tag{1.1}$$

be the Fourier series of a 2π-periodic function $f \in L[-\pi, \pi]$ with an infinity of gaps (n_k, n_{k+1}), where $\{n_k\}$ $(k \in \mathbf{N})$ is a strictly increasing sequence of natural numbers. Noble **[6]** posed the following problem: if the fulfilment of some property of a function f on the whole interval $[-\pi, \pi]$ implies certain conclusions concerning nonlacunary Fourier series $\sigma(f)$ of f, then what lacunae in $\sigma(f)$ guarantees the same conclusions when the property is fulfilled only locally? Several mathematicians including Noble **[6]**, Kennedy **[3]**–**[5]** and Tomić **[10]** and **[11]** studied this problem by considering various properties of f either on an arbitrary subinterval or on an arbitrary subset of positive measure or at an arbitrary point of $[-\pi, \pi]$, and obtained a number of good results under different lacunarity conditions. Proofs of most of these results depend on the fact that

$$a_{n_k} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)P(x) \cos n_k x \, dx$$

(with a similar formula for b_{n_k}) for any trigonometric polynomial $P(x)$ with constant term 1 and of degree less than $\min\{n_k - n_{k-1}, n_{k+1} - n_k\}$. Here $P(x)$ is generally taken to be either the polynomial considered by Noble (e.g.,

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refer: **[6]**, **[4]**, **[5]**) or the Fejér kernel (e.g., refer: **[10]**, **[11]**, **[2]**) or the Jackson kernel (e.g., refer: **[2]**) all of which are very small outside a small subinterval of $[-\pi, \pi]$; thus making $|a_{n_k}|$, $|b_{n_k}|$ to depend mainly on the local behaviour of $f(x)$. This technique is sparingly used in studying the above problem posed by Noble. However, the more powerful methods of approach to this kind of problem, employed by Kennedy **[3]** and originally developed by Paley and Wiener **[7]**, are still not used to their best. In this paper, employing these methods, we have studied the absolute convergence of the lacunary Fourier series (1.1) when the function f satisfies certain hypothesis in terms of either the modulus of continuity or the modulus of smoothness of order l considered only at a point. For this purpose we prove an inequality (see Lemma 4) which is a kind of generalization of the inequality established by Patadia **[8]**.

2. Notations. Let x_0 be an arbitrary fixed point of $[-\pi, \pi]$, $\omega(t)$ ($t \in \mathbf{R}$, $t > 0$) be the modulus of continuity of f at the point x_0 defined by

$$\omega(t) = \sup_{0<|h|<t} \{|f(x_0 + h) - f(x_0)|\} \tag{2.1}$$

and $\omega_l(t)$ be the modulus of smoothness of f of order l ($l \in \mathbf{N}$) at the point x_0 defined by

$$\omega_l(t) = \sup_{0<|h|<t} \left\{ \left| \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x_0 + (2j - l)h) \right| \right\}. \tag{2.2}$$

We consider the lacunarity condition

$$(n_{k+1} - n_k) > CF(n_k), \tag{2.3}$$

where $F(n_k)$ increases to ∞ as $k \to \infty$, $F(n_k) \leqslant n_k$ for all $k \in \mathbf{N}$ and $C > 0$ is a constant. With $F(n_k) = n_k$ this condition gives rise to the Hadamard lacunarity condition

$$\operatorname*{limit\,inf}_{k\to\infty} \frac{n_{k+1}}{n_k} > 1. \tag{2.4}$$

Put $\delta = 8\pi/(CF(n_T))$, where T is a natural number and let

$$I = \{x : |x - x_0| \leqslant \delta\}.$$

3. Statement of the results.

THEOREM 1. *If*

$$\sum_{k=1}^{\infty} \frac{(\omega(A/F(n_k)))^{\beta}}{k^{\beta/2}} < \infty \qquad (0 < \beta \leqslant 1) \tag{3.1}$$

and if $\{n_k\}$ *satisfies* (2.3) *then for the Fourier series* (1.1) *of* $f \in L^2(I)$ (*for some* I) *we have*

$$\sum_{k=1}^{\infty} \left(|a_{n_k}|^{\beta} + |b_{n_k}|^{\beta}\right) < \infty, \tag{3.2}$$

where $\omega(A/F(n_k))$ *is as in* (2.1) *with* t *replaced by* $A/F(n_k)$, $A = 24\pi/C + \pi$.

THEOREM 2. *Theorem* 1 *holds if* (3.1) *is replaced by the condition*

$$\sum_{k=1}^{\infty} \frac{\left(\omega_l(B/F(n_k))\right)^{\beta}}{k^{\beta/2}} < \infty,$$

where $\omega_l(B/F(n_k))$ *is as in* (2.2) *with* t *replaced by* $B/F(n_k)$ *in which* $B = 8\pi/C + \pi$ *and* l *is an odd natural number.*

Observe that the following corollary follows from Theorem 2 if we take $F(n_k) = n_k$.

COROLLARY. *If*

$$\sum_{k=1}^{\infty} \frac{\left(\omega_l(B/n_k)\right)^{\beta}}{k^{\beta/2}} < \infty$$

and if $\{n_k\}$ *satisfies the Hadamard lacunarity condition* (2.4) *then for the Fourier series* (1.1) *of* $f \in L^2(I)$ (*for some* I) *we have* (3.2).

REMARK. With $l = \beta = 1$, without the lacunarity condition and with modulus of continuity $\omega(B/k)$ considered on the whole interval $[-\pi, \pi]$ instead of at the point x_0, the corollary is equivalent to the theorem due to Bernšteĭn [**1**, p. 154].

4. Proofs of the results. We need the following lemmas. Lemma 1 is a special case of a very general theorem due to Paley and Wiener [**7**, Theorem XLII′]. Lemma 2 is due to Stečkin [**9**, Lemma 2]. The inequality (4.2) of Lemma 4 is a simple consequence of the more general lemma quoted by Kennedy [**3**, Lemma 1].

LEMMA 1. *If* $f \in L^2(I_1)$, *where* I_1 *is an interval, and if* $(n_{k+1} - n_k) \to \infty$ *in* (1.1) *then* $f \in L^2[-\pi, \pi]$.

LEMMA 2. *If* $u_n \geqslant 0$ $(n \in \mathbf{N})$, $u_n \not\equiv 0$ *and* $F(u_n)$ *is a function such that* $F(0) = 0$, $F(u)$ *increasing and concave, then*

$$\sum_{n=1}^{\infty} F(u_n) < 2 \sum_{n=1}^{\infty} F\left(\frac{u_n + u_{n+1} + \cdots\cdots\cdots}{n}\right).$$

LEMMA 3. *The modulus of continuity* $\omega(t)$ *defined as in* (2.1) *is such that* $\omega(0) = 0$, $\omega(t) > 0$ *and* $\omega(t)$ *is increasing.*

PROOF. Obvious.

LEMMA 4. *Let* $f \in L^2(I)$ *for some* I, $n_0 = 0$, $n_k = -n_{-k}$ $(k < 0)$; $C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$ *and* $C_{n_k} = \overline{C}_{n_{-k}}$ $(k < 0)$. *If*

$$(n_{k+1} - n_k) \geqslant 8\pi\delta^{-1} \quad \textit{for all } k \tag{4.1}$$

then

$$\sum_{-\infty}^{\infty} |C_{n_k}|^2 \leqslant 8\delta^{-1} \int_I |f(x)|^2\, dx, \tag{4.2}$$

$$\sum_{|n_k| > n_T} |C_{n_k}|^2 \leqslant D\left(\omega(A/F(n_T))\right)^2 \tag{4.3}$$

or more generally

$$\sum_{|n_k| > n_T} |C_{n_k}|^2 \leqslant D\left(\omega_l(B/F(n_T))\right)^2, \tag{4.4}$$

where D is some constant and l is an odd natural number.

PROOF. We have

$$\sum_{-\infty}^{\infty} |C_{n_k}| r^{|n_k|} < \infty \qquad (0 < r < 1) \tag{4.5}$$

and if we put

$$\phi(r, x) = \sum_{-\infty}^{\infty} C_{n_k} r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1) \tag{4.6}$$

for all real x, then its existence is assured by (4.5) and we obviously get

$$\phi(r, x) = \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x) r^{n_k}. \tag{4.7}$$

Since $f \in L^2(I)$ and (2.3) holds therefore $f \in L^2[-\pi, \pi]$ by Lemma 1. Hence by a known theorem [**12**, p. 87] it follows that

$$\operatorname*{limit}_{r \to 1} \int_{-\pi}^{\pi} |\phi(r, x) - f(x)|^2 \, dx = 0. \tag{4.8}$$

Now, on account of (4.1), (4.5), (4.6) and (4.8), we can apply the lemma quoted in [**3**, Lemma 1] to obtain the inequality (4.2).

Now put

$$g(x) = f(x + h) - f(x - h) \tag{4.9}$$

and

$$C^*_{n_k} = 2i\, C_{n_k} \sin n_k h. \tag{4.10}$$

Then $|C^*_{n_k}| \leqslant 2|C_{n_k}|$ and hence by (4.5) we have

$$\sum_{-\infty}^{\infty} |C^*_{n_k}| r^{|n_k|} < \infty \qquad (0 < r < 1). \tag{4.11}$$

Put

$$g(r, x) = \sum_{-\infty}^{\infty} C^*_{n_k} r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1). \tag{4.12}$$

We then get the identity $g(r, x) = \phi(r, x + h) - f(r, x - h)$ and from it together with (4.8) and (4.9) we obtain

$$\operatorname*{limit}_{r \to 1} \int_{-\pi}^{\pi} |g(r, x) - g(x)|^2 \, dx = 0. \tag{4.13}$$

It follows from (4.1), (4.11), (4.12) and (4.13) and the inequality (4.2) that

$$\sum_{-\infty}^{\infty} |C^*_{n_k}|^2 \leqslant 8\delta^{-1} \int_I |g(x)|^2 \, dx. \tag{4.14}$$

Hence by (4.9) and (4.10) we get

$$4 \sum_{-\infty}^{\infty} |C_{n_k}|^2 \sin^2 |n_k| h \leqslant 8\delta^{-1} \int_I |f(x+h) - f(x-h)|^2 \, dx. \quad (4.15)$$

Integrating both the sides of (4.15) with respect to h over $(0, \pi/n_T)$ we get

$$4 \sum_{-\infty}^{\infty} |C_{n_k}|^2 \int_0^{\pi/n_T} \sin^2 |n_k| h \, dh$$

$$\leqslant 8 \frac{CF(n_T)}{8\pi} \int_0^{\pi/n_T} dh \left(\int_{x_0 - 8\pi/(CF(n_T))}^{x_0 + 8\pi/(CF(n_T))} |f(x+h) - f(x-h)|^2 \, dx \right). \quad (4.16)$$

We see that if $|n_k| \geqslant n_T$ then

$$\int_0^{\pi/n_T} \sin^2 |n_k| h \, dh = \frac{1}{|n_k|} \int_0^{(|n_k|/n_T)\pi} \sin^2 t \, dt$$

$$> \frac{1}{n_T([|n_k|/n_T] + 1)} \int_0^{[|n_k|/n_T]\pi} \sin^2 t \, dt$$

$$= \frac{1}{n_T} \frac{[|n_k|/n_T]}{1 + [|n_k|/n_T]} \frac{\pi}{2}$$

$$\geqslant \frac{1}{n_T} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4n_T}, \quad (4.17)$$

using $1 \leqslant [|n_k|/n_T] \leqslant |n_k|/n_T \leqslant 1 + [|n_k|/n_T]$, where [ ] denotes the integral part.

Also, by Lemma 3, using $\pi/n_T \leqslant \pi/F(n_T)$ and observing from (4.16) that $x \in [x_0 - \delta, x_0 + \delta]$, we obtain

$$|f(x+h) - f(x-h)|$$

$$= |f(x_0 - \delta + \eta + h) - f(x_0) + f(x_0) - f(x_0 - \delta + \eta - h)| \quad (0 \leqslant \eta \leqslant 2\delta)$$

$$\leqslant 2\omega(\delta + \eta + h) \leqslant 2\omega(3\delta + h)$$

$$\leqslant 2\omega\big(24\pi/(CF(n_T)) + \pi/n_T\big)$$

$$\leqslant 2\omega\big(24\pi/(CF(n_T)) + \pi/F(n_T)\big)$$

$$= 2\omega(A/F(n_T)), \qquad A = \pi + 24\pi/C. \quad (4.18)$$

Using (4.17) and (4.18), we get from (4.16)

$$\frac{\pi}{n_T} \sum_{|n_k| \geqslant n_T} |C_{n_k}|^2 \leqslant 8 \frac{CF(n_T)}{8\pi} \frac{\pi}{n_T} \frac{16\pi}{CF(n_T)} 4\big(\omega(A/F(n_T))\big)^2.$$

This gives the inequality (4.3). Further, if we put

$$g(x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x + (2j - l)h) \quad (4.9)'$$

and

$$C^*_{n_k} = C_{n_k} \exp(-in_k lh)(\exp(2in_k h) - 1)^l$$
$$= 2^l C_{n_k} \exp(-iln_k h)(-1)^l \exp(il(n_k h - \pi/2)) \sin^l n_k h, \quad (4.10)'$$

then $|C^*_{n_k}| \leqslant 2^l |C_{n_k}|$ and hence by (4.5) we have

$$\sum_{-\infty}^{\infty} |C^*_{n_k}| r^{|n_k|} < \infty \qquad (0 < r < 1). \qquad (4.11)'$$

Put

$$g(r, x) = \sum_{-\infty}^{\infty} C^*_{n_k} r^{|n_k|} \exp(in_k x) \qquad (0 < r < 1). \qquad (4.12)'$$

We shall then get the identity

$$g(r, x) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} \phi(r, x + (2j - l)h).$$

This together with (4.8) and (4.9) gives

$$\operatorname*{limit}_{r \to 1} \int_{-\pi}^{\pi} |g(r, x) - g(x)|^2 \, dx = 0. \qquad (4.13)'$$

It follows from (4.1), (4.11)′, (4.12)′, (4.13)′ and the inequality (4.2) that

$$\sum_{-\infty}^{\infty} |C^*_{n_k}|^2 \leqslant 8\delta^{-1} \int_I |g(x)|^2 \, dx. \qquad (4.14)'$$

Again, as in the foregoing proof of (4.3), analogously we get

$$\int_0^{\pi/n_T} \sin^{2l} |n_k| h \, dh \geqslant \frac{1}{n_T} \; \frac{[|n_k|/n_T]}{(1 + [|n_k|/n_T])} \int_0^{\pi} \sin^{2l} t \, dt$$
$$> \frac{1}{n_T} \cdot \frac{1}{2} \cdot \frac{2l-1}{2l} \cdot \frac{2l-3}{2l-2} \cdots \frac{1}{2} \pi$$
$$> \frac{\pi}{2^{l+1} n_T}. \qquad (4.17)'$$

Also, proceeding analogously as in the proof of (4.18) and observing that l is an odd natural number, we get

$$\left| \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x + (2j - l)h) \right|$$
$$= \left| \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x_0 + (2j - l)(h + (\eta - \delta)/(2j - l))) \right|$$
$$\leqslant \omega_l(h + \eta - \delta)$$
$$\leqslant \omega_l(h + \delta)$$
$$\leqslant \omega_l(B/F(n_T)), \qquad B = 8\pi/C + \pi.$$

Using this along with (4.17)′ and (4.14)′ and proceeding analogously as in the proof of (4.3), (4.4) is proved. This completes the proof of Lemma 4.

PROOF OF THEOREM 1. Let $n_0 = 0$, $n_k = -n_{-k}$ $(k < 0)$; $C_{n_0} = 0$, $C_{n_k} = \frac{1}{2}(a_{n_k} - ib_{n_k})$ $(k > 0)$, $C_{n_k} = \bar{C}_{n_{-k}}$ $(k < 0)$. We assume without loss of generality that (4.1) holds. In view of (2.3) this can be achieved, if necessary, by adding to $f(x)$ a polynomial in $\exp(in_k x)$, a process which affects neither different T, this polynomial may of course be different.) Then Lemma 4 holds and putting $r_{n_T} = \sum_{|n_k| \geqslant n_T} |C_{n_k}|^2$ in the inequality (4.3) we get

$$r_{n_T}^{\beta/2} \leqslant C\big(\omega(A/F(n_T))\big)^{\beta}, \tag{4.19}$$

where C is a constant.

Now applying the Lemma 2 with $u_k = |C_{n_k}|^2$ $(k \in \mathbf{Z})$ and $F(u) = u^{\beta/2}$, we obtain using (4.19)

$$\sum_{-\infty}^{\infty} |C_{n_k}|^{\beta} = 2 \sum_{k=1}^{\infty} F\big(|C_{n_k}|^2\big) \leqslant 4 \sum_{k=1}^{\infty} \big(r_{n_k}/k\big)^{\beta/2}$$

$$\leqslant 4C \sum_{k=1}^{\infty} \Big(\big(\omega(A/F(n_k))\big)^{\beta}/k^{\beta/2}\Big) < \infty$$

on account of (3.1). Therefore (3.2) holds and this completes the proof of the theorem.

PROOF OF THEOREM 2. Applying the inequality (4.4) instead of (4.3) in the proof of Theorem 1 and proceeding analogously, this theorem is proved.

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