PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

EULER CHARACTERSITICS AND CODIMENSIONS OF COMPLETE INTERSECTIONS

BANG-YEN CHEN

ABSTRACT. Studies on relations between Euler characteristics and codimensions of complete intersections.

Let $F_1, F_2, \ldots, F_r$ be nonsingular hypersurfaces of degrees $a_1, a_2, \ldots, a_r$, in complex projective space CP^{n+r}, and suppose that these hypersurfaces are in general position. The intersection $F_1 \cap F_2 \cap \cdots \cap F_r$ is a nonsingular algebraic variety denoted by $V_n[a_1, \ldots, a_r]$. In this short note, we prove the following theorem which completes the solution to the problem studied in [**1**]. The presentation of the proof follows closely that of the proofs in [**1**].

THEOREM. *Let V_n be an n-dimensional complete intersection with Euler characteristic $\chi(V_n) = \nu_1 \cdots \nu_p$ for some prime numbers $\nu_1, \ldots, \nu_p$ $(\neq \pm 1)$. Then V_n can be imbedded in CP^{n+p-1} as a complete intersection except when V_n is $V_1[2]$ or $V_1[2, 3]$ or $V_1[2, 2, 2]$.*

PROOF. In [**2**], Hirzebruch proved the following identity:

$$\sum_{n=0}^{\infty} \chi(V_n[a_1, a_2, \ldots, a_r])z^n = \frac{a_1 a_2 \cdots a_r}{(1-z)^2} \prod_{i=1}^{r} \frac{1}{1+(a_i-1)z}. \tag{1}$$

By multiplying power series, (1) implies

$$\chi(V_n[a]) = \frac{(1-a)^{n+2} - 1 + (n+2)a}{a^2} \cdot a, \tag{2}$$

$$\begin{aligned}(-1)^n \chi(V_n[a_1, a_2, \ldots, a_r]) \\ = a_r \sum_{k=0}^{n} (a_r - 1)^{n-k}(-1)^k \chi(V_k[a_1, a_2, \ldots, a_{r-1}]).\end{aligned} \tag{3}$$

By induction, we find

$$(-1)^n \chi(V_n[a_1, a_2, \ldots, a_r]) = a_1 a_2 \cdots a_r h_n[a_1, a_2, \ldots, a_r], \tag{4}$$

where

Received by the editors June 20, 1977.

AMS (MOS) subject classifications (1970). Primary 14M10, 57D20.

Key words and phrases. Euler characteristic, complete intersection, codimension.

$$h_n[a_1, a_2, \ldots, a_r] = \sum_{k_r=0}^{n} \sum_{k_{r-1}=0}^{k_r} \cdots \sum_{k_2=0}^{k_3} (-1)^{k_2}(a_r - 1)^{n-k_r} \cdots (a_2 - 1)^{k_3-k_2}\Delta_{k_2}[a_i],$$

and $\Delta_k[a] = \{(1 - a)^{k+2} - 1 + (k + 2)a\}/a^2$. It is clear that $h_n[a] = (-1)^n\Delta_n[a]$, $\Delta_1[3] = 0$, $\Delta_0[a] = 1$, $\Delta_n[2] = (n + 2)/2$ when n is even, $\Delta_n[2] = (n + 1)/2$ when n is odd, and $(-1)^n\Delta_n[a] > 0$ when $n \geqslant 2$ and $a \geqslant 3$. Thus, we obtain

$$\begin{gathered} h_n[a_1, a_2, \ldots, a_r] > 1 \text{ for } r \geqslant 2, \text{ except } n = 1, r = 2 \\ a_1 \leqslant 2, a_2 \leqslant 3, \text{ or } n = 1, r = 3, a_1, a_2, a_3 \leqslant 2. \end{gathered} \tag{5}$$

$$h_n[a] \neq \pm 1 \quad \text{except } n = 1, a = 2 \text{ or } 4. \tag{6}$$

Now, we assume that $V_n[a_1, a_2, \ldots, a_r]$; $a_1, a_2, \ldots, a_r \geqslant 2$, is a complete intersection with $\chi(V_n) = \nu_1 \cdots \nu_p$ for some prime integers $\nu_1, \ldots, \nu_p \neq \pm 1$. If $r < p$, then it is done. If $r \geqslant p$, then (4) implies $r = p$ and $h_n[a_1, a_2, \ldots, a_r] = \pm 1$. From (5) and (6) we see that this is impossible unless $n = 1$ and V_n is one of the following: $V_1[2, 2]$, $V_1[2, 3]$, $V_1[2, 2, 2]$, $V_1[2]$ or $V_1[4]$. Since we have $\chi(V_1[2, 2]) = 0$, $\chi(V_1[2, 3]) = -6$, $\chi(V_1[2, 2, 2]) = -8$, $\chi(V_1[2]) = 2$ and $\chi(V_1[4]) = -4$, the theorem is proved.

References

1. B.-Y. Chen and K. Ogiue, *Some implications of the Euler-Poincaré characteristic for complete intersection manifolds*, Proc. Amer. Math. Soc. **44** (1974), 1–8.

2. F. Hirzebruch, *Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung einige Anwendungen und offene Fragen*, Proc. Internat. Congress Math. (Amsterdam, 1954), Vol. III, Noordhoff, Groningen; Amsterdam, 1956, pp. 457–473.

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824