

EULER CHARACTERISTICS AND CODIMENSIONS OF COMPLETE INTERSECTIONS

BANG-YEN CHEN

ABSTRACT. Studies on relations between Euler characteristics and codimensions of complete intersections.

Let F_1, F_2, \dots, F_r be nonsingular hypersurfaces of degrees a_1, a_2, \dots, a_r , in complex projective space CP^{n+r} , and suppose that these hypersurfaces are in general position. The intersection $F_1 \cap F_2 \cap \dots \cap F_r$ is a nonsingular algebraic variety denoted by $V_n[a_1, \dots, a_r]$. In this short note, we prove the following theorem which completes the solution to the problem studied in [1]. The presentation of the proof follows closely that of the proofs in [1].

THEOREM. *Let V_n be an n -dimensional complete intersection with Euler characteristic $\chi(V_n) = v_1 \cdots v_p$ for some prime numbers v_1, \dots, v_p ($\neq \pm 1$). Then V_n can be imbedded in CP^{n+p-1} as a complete intersection except when V_n is $V_1[2]$ or $V_1[2, 3]$ or $V_1[2, 2, 2]$.*

PROOF. In [2], Hirzebruch proved the following identity:

$$\sum_{n=0}^{\infty} \chi(V_n[a_1, a_2, \dots, a_r])z^n = \frac{a_1 a_2 \cdots a_r}{(1-z)^2} \prod_{i=1}^r \frac{1}{1+(a_i-1)z}. \quad (1)$$

By multiplying power series, (1) implies

$$\chi(V_n[a]) = \frac{(1-a)^{n+2} - 1 + (n+2)a}{a^2} \cdot a, \quad (2)$$

$$\begin{aligned} & (-1)^n \chi(V_n[a_1, a_2, \dots, a_r]) \\ &= a_r \sum_{k=0}^n (a_r - 1)^{n-k} (-1)^k \chi(V_k[a_1, a_2, \dots, a_{r-1}]). \end{aligned} \quad (3)$$

By induction, we find

$$(-1)^n \chi(V_n[a_1, a_2, \dots, a_r]) = a_1 a_2 \cdots a_r h_n[a_1, a_2, \dots, a_r], \quad (4)$$

where

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$$h_n[a_1, a_2, \dots, a_r] = \sum_{k_r=0}^n \sum_{k_{r-1}=0}^{k_r} \cdots \sum_{k_2=0}^{k_3} (-1)^{k_2} (a_r - 1)^{n-k_r} \cdots (a_2 - 1)^{k_3 - k_2} \Delta_{k_2}[a_i],$$

and $\Delta_k[a] = \{(1-a)^{k+2} - 1 + (k+2)a\}/a^2$. It is clear that $h_n[a] = (-1)^n \Delta_n[a]$, $\Delta_1[3] = 0$, $\Delta_0[a] = 1$, $\Delta_n[2] = (n+2)/2$ when n is even, $\Delta_n[2] = (n+1)/2$ when n is odd, and $(-1)^n \Delta_n[a] > 0$ when $n \geq 2$ and $a \geq 3$. Thus, we obtain

$$h_n[a_1, a_2, \dots, a_r] > 1 \text{ for } r \geq 2, \text{ except } n = 1, r = 2 \quad (5)$$

$$a_1 \leq 2, a_2 \leq 3, \text{ or } n = 1, r = 3, a_1, a_2, a_3 \leq 2.$$

$$h_n[a] \neq \pm 1 \text{ except } n = 1, a = 2 \text{ or } 4. \quad (6)$$

Now, we assume that $V_n[a_1, a_2, \dots, a_r]$; $a_1, a_2, \dots, a_r \geq 2$, is a complete intersection with $\chi(V_n) = \nu_1 \cdots \nu_p$ for some prime integers $\nu_1, \dots, \nu_p \neq \pm 1$. If $r < p$, then it is done. If $r \geq p$, then (4) implies $r = p$ and $h_n[a_1, a_2, \dots, a_r] = \pm 1$. From (5) and (6) we see that this is impossible unless $n = 1$ and V_n is one of the following: $V_1[2, 2]$, $V_1[2, 3]$, $V_1[2, 2, 2]$, $V_1[2]$ or $V_1[4]$. Since we have $\chi(V_1[2, 2]) = 0$, $\chi(V_1[2, 3]) = -6$, $\chi(V_1[2, 2, 2]) = -8$, $\chi(V_1[2]) = 2$ and $\chi(V_1[4]) = -4$, the theorem is proved.

REFERENCES

1. B.-Y. Chen and K. Ogiue, *Some implications of the Euler-Poincaré characteristic for complete intersection manifolds*, Proc. Amer. Math. Soc. **44** (1974), 1-8.
2. F. Hirzebruch, *Der Satz von Riemann-Roch in Faisceau-theoretischer Formulierung einige Anwendungen und offene Fragen*, Proc. Internat. Congress Math. (Amsterdam, 1954), Vol. III, Noordhoff, Groningen; Amsterdam, 1956, pp. 457-473.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824