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ON GROUPOIDS DEFINED BY COMMUTATORS

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ABSTRACT. We study matrices R, L which count the numbers of solutions of $ix = j$ and $xi = j$. For slight generalizations of R, L, the relation $RL = LR$ characterizes associativity of a groupoid. For groupoids defined by group commutators $xyx^{-1}y^{-1}$ the relation $RL = LR$ is valid. In addition one can study analogues of Green's relations. Any $\mathcal{J}$-class contains at most four $\mathcal{H}$-classes in a commutator groupoid.

In this paper we mainly consider groupoids whose underlying set is a group, with groupoid multiplication $x * y = xyx^{-1}y^{-1}$. Our interest is mainly in the matrices R and L such that r_{ij} counts the number of solutions of $i * x = j$ and l_{ij} counts the number of solutions of $x * i = j$.

DEFINITION. Let G be a groupoid. Let t, u be functions from G to a commutative semiring K with 0. Then $R(t)$ is the matrix (r_{ij}) for $i, j \in G$ such that $r_{ij} = \Sigma t(x)$, the summation being over all x such that $ix = j$, if this sum is defined. And $L(u)$ is the matrix (l_{ij}) such that $l_{ij} = \Sigma\, u(x)$, the summation being over all x such that $xi = j$ if this sum is defined. Summations over the empty set are considered to be 0. And we assume $0 + k = k$ and $0k = 0$ for all $k \in K$.

In this paper we consider the two cases: (i) G finite, $K = \mathbf{Z}^+ \cup \{0\}$; (ii) G arbitrary, K the Boolean algebra $\{0, 1\}$. The following proposition is essentially due to M. S. Putcha [2].

PROPOSITION 1. *In the two cases just mentioned, the matrices $R(t)$, $L(u)$ commute for all t, u if and only if G is associative.*

PROOF. We have

$$(R(t)L(u))_{ij} = \sum t(x)u(y)$$

where the summation is over all pairs such that $ix = k, yk = j$ for some k, i.e. all pairs such that $y(ix) = j$. Likewise

$$(L(u)R(t))_{ij} = \sum u(y)t(x)$$

where the summation is over all pairs such that $(yi)x = j$. So if G is

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associative $R(t)$, $L(u)$ commute. For the converse, let u, t range independently over all functions which send every element of G except one, to zero. This proves the proposition.

REMARK. By choosing t, u to send elements of G to randomly chosen real numbers, this might give a quick computer test for nonassociativity of a groupoid.

From here on, we assume both t, u send all elements of G to 1, and we write R, L for $R(t)$, $L(u)$.

DEFINITION. A *group commutator groupoid* is a groupoid G whose underlying set is a group and whose groupoid product is given by $xyx^{-1}y^{-1}$.

PROPOSITION 2. *Let G be a group commutator groupoid. Let T be the matrix of the permutation $x \to x^{-1}$. Then $RT = TR = L$. Therefore R, L commute.*

PROOF. The equation $RT = L$ follows from $(ixi^{-1}x^{-1})^{-1} = xix^{-1}i^{-1}$. The identity $i^{-1}xix^{-1} = (i^{-1}xi)i(i^{-1}xi)^{-1}i^{-1}$ implies $TR = RT$.

PROPOSITION 3. *For each a, b, R_{ab} and L_{ab} are each either zero or the order of the centralizer of a. The row sums of R, L all equal the order of G. The bth column sum of R and the bth column sum of L each equal the number of pairs x, y such that $xyx^{-1}y^{-1} = b$. The trace of R equals the order of G. The trace of L equals the sums of the orders of the centralizers of those elements a which are conjugate to a^2.*

PROOF. The entry R_{ab} is the number of solutions of $xa^{-1}x^{-1} = a^{-1}b$. This is either zero or has the same order as the centralizer of a^{-1}. But the centralizer of a equals the centralizer of a^{-1}. Likewise for L_{ab}. The second and third statements can be observed to be true. For the fourth statement, note that the trace of R is the sum of the orders of the centralizers of such that $xa^{-1}x^{-1} = e$. But this can happen only if $a = e$. Likewise for L. This proves the proposition.

DEFINITION. A (*left*, *right*) *ideal* in a groupoid is a subset closed under (left, right) multiplication. The principal (left, right) ideal generated by an element is the intersection of all (left, right) ideals containing that element. Two elements are ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$)-*equivalent* if and only if they generate the same principal (right, left, two-sided) ideal. They are $\mathcal{H}$-*equivalent* if and only if they are both $\mathcal{R}$- and $\mathcal{L}$-equivalent. These equivalence relations are called *Green's relations*.

DEFINITION. A directed graph is *strongly connected* if and only if every point can be reached from every other point by a directed path.

Corresponding to this one can express any graph as a disjoint union of its strong components. We consider the graph of a matrix to be the graph whose vertices are the elements of the index set of the matrix, having an edge from i to j if and only if the (i, j)-entry of the matrix is nonzero.

PROPOSITION 4. *For any groupoid, the strong components of the graphs of*

$I + R$, $I + L$, $(I + R)(I + L)$ are the $\mathfrak{R}$, $\mathfrak{L}$, $\mathfrak{J}$-classes. Here I denotes the identity matrix.

Note that if the elements of G are arranged in the order of an ascending chain of normal subgroups, the matrices R, L will assume a block triangular form. In addition nilpotency can easily be detected.

THEOREM 5. *A finite group G is nilpotent if and only if the matrix R of its commutator groupoid is nilpotent. Likewise for L.*

PROOF. Suppose G is nilpotent. Arrange the elements of G in the order of an ascending central series. Then R, L are lower subtriangular matrices.

Suppose G is not nilpotent. Then by Theorem 14.4.7 of [1] there exist x, p such that x has order prime to p and x normalizes but does not centralize some p subgroup Q. Choose Q to be minimal. Then x acts trivially on $[Q, Q]$ by conjugation. Then x does not act trivially on $Q/[Q, Q]$ by conjugation, or the group generated by x, Q would have a central series. So x gives a nontrivial automorphism of $Q/[Q, Q]$. An endomorphism of $Q/[Q, Q]$ is given by $y \to xyx^{-1}y^{-1}$, mod$[Q, Q]$. If this endomorphism were nilpotent, the automorphism xyx^{-1} would have order a power of p, which is false. Thus the endomorphism of $Q/[Q, Q]$ given by $y \to xyx^{-1}y^{-1}$ is not nilpotent. This implies L is nonnilpotent. Similarly for R.

THEOREM 6. *If G is a group commutator groupoid, every $\mathfrak{J}$-class of G contains at most two $\mathfrak{R}$-classes and at most two $\mathfrak{L}$-classes. If there are two of either type, they are equal in size. And $a \mathfrak{J} b$ if and only if there exists c such that $a \mathfrak{R} c$, $c \mathfrak{L} b$ if and only if there exists d such that $a \mathfrak{L} d$, $d \mathfrak{R} b$.*

PROOF. The classes will not be affected if we use matrices over the Boolean algebra $\{0, 1\}$ always, The classes obtained from $I + R$, $I + L$, $(I + R)(I + L)$ are the same as those obtained from

$$\bar{R} = I + R + R^2 + \dots,$$
$$\bar{L} = I + L + L^2 + \dots,$$
$$\bar{R}\bar{L} = \sum_0^\infty R^n + \sum_1^\infty R^n T.$$

Suppose $a \mathfrak{J} b$. Note that $\bar{R}$, $\bar{L}$, $\bar{R}\bar{L}$ are idempotent. Thus there is an edge in the graph of $\bar{R}\bar{L}$ from a to b and one from b to a. Each of these two edges comes from one of the two summands

$$\sum_0^\infty R^n, \qquad \sum_1^\infty R^n T.$$

In the first case there is an $\bar{R}$ edge from one to the other and in the second case there is an $\bar{R}$ edge from one to the inverse of the other. We denote the existence of an edge from one to the other by $\to$. We observe that $x \to y^{-1}$ if and only if $x^{-1} \to y$ since $RT = TR$. There are four cases:

Case 1. $a \to b$, $b \to a$ in the graph of $\bar{R}$. Then $a \mathfrak{R} b$.

Case 2. $a \to b^{-1}$, $b \to a^{-1}$ in the graph of $\bar{R}$. Then $a\ \mathfrak{R}\ b^{-1}$.

Case 3. $a \to b$, $b \to a^{-1}$ in the graph of $\bar{R}$. Then also $a^{-1} \to b^{-1}$, $b^{-1} \to a$. These imply $a\ \mathfrak{R}\ b$.

Case 4. $a \to b^{-1}$, $b \to a$. Again $a\ \mathfrak{R}\ b$. Therefore either a lies in the $\mathfrak{R}$-class of b or that of b^{-1}. Thus the $\mathfrak{J}$-class of b contains at most two $\mathfrak{R}$-classes. Likewise it contains at most two $\mathfrak{L}$-classes.

Suppose there do exist two $\mathfrak{R}$-classes in some $\mathfrak{J}$-class. Then there exist a, b such that $a\ \mathfrak{J}\ b$ but not $a\ \mathfrak{R}\ b$. Thus the situation must be that of Case 2. And for any a, b in different $\mathfrak{R}$-classes but in the same $\mathfrak{J}$-class, this must be so. Therefore $a\ \mathfrak{R}\ b^{-1}$. Thus for any b in this $\mathfrak{J}$-class, b and b^{-1} will lie in different $\mathfrak{R}$-classes. Therefore the mapping $x \to x^{-1}$ will be a 1-1 onto mapping from one $\mathfrak{R}$-class to the other. Likewise for $\mathfrak{L}$-classes.

In Cases 1, 3, 4, $a\ \mathfrak{R}\ b$ and the last statement is valid. Suppose we are in the second case. Suppose $a \to b^{-1}$ by an odd number of edges in the graph of R, and $b^{-1} \to a$ by an odd number. Then since $L = RT$, $a\ \mathfrak{L}\ b$. Suppose $a \to b^{-1}$ by an even number of $\mathfrak{R}$ edges and $b^{-1} \to a$ by an even number. Let $a \to x$ be the first edge in the sequence from a to b^{-1}. Then $a \to x \to b^{-1} \to a \to x$. So $a\ \mathfrak{R}\ x$, $x\ \mathfrak{L}\ b$. And $a\ \mathfrak{L}\ x^{-1}$, $x^{-1}\ \mathfrak{R}\ b$. If the number of edges from a to b^{-1} is even and the number of edges from b^{-1} to a is odd, or vice versa, we can double the path and obtain one of the two former cases. This proves the theorem.

Example 1. For the symmetric group on three symbols, L and R are, respectively

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2. It is difficult to find a finite group with a $\mathfrak{J}$-class containing four different $\mathfrak{H}$-classes. Consider the semidirect product of the multiplicative group of numbers of the form $\pi^i(\pi - 1)^j$ with the additive real numbers. Then $1\ \mathfrak{L}\ \pi - 1$ but 1 and $\pi - 1$ are not $\mathfrak{R}$-equivalent. Also $1\ \mathfrak{R}\ 1 - \pi$ but 1 and $1 - \pi$ are not $\mathfrak{L}$-equivalent. Then Theorem 6 implies there are at least four distinct $\mathfrak{H}$-classes, in the $\mathfrak{J}$-class of 1.

Remark. Many of the results demonstrated here are trivially true for groupoids defined by Lie algebra commutators.

References

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