

## ASCENT, DESCENT AND COMPACT PERTURBATIONS

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**ABSTRACT.** The collections of upper semi-Fredholm operators with finite ascent and of lower semi-Fredholm operators with finite descent are both closed under commuting compact perturbations.

Suppose that  $T$  and  $V$  are commuting bounded linear operators on the Banach space  $X$  and that  $T - V$  is compact. In Theorem 2 below we show that if  $T$  is upper semi-Fredholm, then  $T$  has finite ascent if and only if  $V$  does; and, dually, if  $T$  is lower semi-Fredholm, it has finite descent if and only if  $V$  has finite descent. If  $T$  is invertible or just Fredholm of index 0, it has long been known that  $V$  has finite ascent and descent [3, Theorem 6.3, p. 610], [1, Theorem (1.4.5), p. 12], [2, pp. 39–42]; but, even in this special case, examples show that the commutativity of  $T$  and  $V$  is crucial [3, p. 599], [1, pp. 13–14], [2, p. 40].

We start with a lemma which treats the special case that  $T$  is onto.

**LEMMA 1.** *Suppose that  $T$  and  $V$  are commuting bounded linear operators on the Banach space  $X$ . If  $T - V$  is compact and  $T$  is onto, then  $V$  has finite descent.*

**PROOF.** For each nonnegative integer  $k$ , the range,  $R(V^k)$ , has finite codimension [1, Corollary (1.3.7)(b), p. 9] and the map induced by  $T$  on  $X/R(V^k)$  is onto. Therefore this induced map is one-to-one, so that the null-space  $N(T) \subseteq R(V^k)$ . Since  $T$  is onto, there is a positive number  $\gamma$  for which  $\|Tx\| \geq \gamma \operatorname{dist}(x, N(T))$  for all  $x$  in  $X$ . Suppose that  $x$  belongs to  $X$  and  $z$  belongs to  $R(V^k)$ ; then  $T(R(V^k)) = R(V^kT) = R(V^k)$  so there is a  $y$  in  $R(V^k)$  with  $Ty = z$ . Thus we have  $\|Tx - z\| = \|T(x - y)\| \geq \gamma \operatorname{dist}(x - y, N(T)) \geq \gamma \operatorname{dist}(x, R(V^k))$ , since  $N(T) \subseteq R(V^k)$ . Since this holds for all  $z$  in  $R(V^k)$ , we obtain

$$\operatorname{dist}(Tx, R(V^k)) \geq \gamma \operatorname{dist}(x, R(V^k)).$$

Suppose  $V$  had infinite descent. Then there would be a bounded sequence  $\{x_n\}$  with  $x_n \in R(V^n)$  and  $\operatorname{dist}(x_n, R(V^{n+1})) \geq 1$ . Let  $K = T - V$  and suppose  $m > n$ . Then  $Kx_m - Kx_n = (Kx_m + (T - K)x_n) - Tx_n$ . So that

$$\|Kx_m - Kx_n\| \geq \operatorname{dist}(Tx_n, R(V^{n+1})) \geq \gamma \operatorname{dist}(x_n, R(V^{n+1})) \geq \gamma.$$

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But this contradicts the compactness of  $K$ , so  $V$  must have finite descent.

**THEOREM 2.** *Suppose that  $T$  and  $V$  are commuting bounded linear operators on the Banach space  $X$  with  $T - V$  compact.*

(A) *If  $T$  is upper semi-Fredholm, then  $V$  has finite ascent if and only if  $T$  has finite ascent.*

(B) *If  $T$  is lower semi-Fredholm, then  $V$  has finite descent if and only if  $T$  has finite descent.*

**PROOF.** Suppose first that  $T$  is lower semi-Fredholm. Since  $V$  is also lower semi-Fredholm [1, Corollary (1.3.7)(b), p. 9], it will be enough to show that  $V$  has finite descent if  $T$  has. Let  $p$  be an integer with  $R(T^p) = R(T^{p+1})$ . Then  $R(T^p)$  is a closed subspace of finite codimension [1, Corollary (1.3.3), p. 9] and the restriction of  $T$  to  $R(T^p)$  is onto. Therefore, by Lemma 1, the restriction of  $V$  to  $R(T^p)$  has finite descent, so that there is an integer  $k$  for which

$$R(V^m) \supseteq R(V^m T^p) = R(V^k T^p)$$

for all  $m \geq k$ . Since  $R(V^k T^p)$  has finite codimension [1, Corollary (1.3.3), p. 9],  $V$  has finite descent. This proves (B).

Now suppose that  $T$  and  $V$  are upper semi-Fredholm. Then  $T^*$  and  $V^*$  are lower semi-Fredholm, and the ascent of  $T$  and  $V$ , respectively, equals the descent of  $T^*$  and of  $V^*$ , respectively [1, pp. 7-8]. Part (A) now follows directly from Part (B).

Instead of proving Part (A) from Part (B) in Theorem 2 by duality arguments, we could have proved the dual result to Lemma 1 for  $T$  bounded below and then used this result to prove Part (A). The direct proofs are very similar to our proofs of Lemma 1 and Theorem 2(B).

In subsequent papers we will use the results of the present paper to study compact perturbations of more general classes of operators.

#### REFERENCES

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