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ASCENT, DESCENT AND COMPACT PERTURBATIONS

SANDY GRABINER

ABSTRACT. The collections of upper semi-Fredholm operators with finite ascent and of lower semi-Fredholm operators with finite descent are both closed under commuting compact perturbations.

Suppose that T and V are commuting bounded linear operators on the Banach space X and that $T - V$ is compact. In Theorem 2 below we show that if T is upper semi-Fredholm, then T has finite ascent if and only if V does; and, dually, if T is lower semi-Fredholm, it has finite descent if and only if V has finite descent. If T is invertible or just Fredholm of index 0, it has long been known that V has finite ascent and descent [**3**, Theorem 6.3, p. 610], [**1**, Theorem (1.4.5), p. 12], [**2**, pp. 39–42]; but, even in this special case, examples show that the commutativity of T and V is crucial [**3**, p. 599], [**1**, pp. 13–14], [**2**, p. 40].

We start with a lemma which treats the special case that T is onto.

LEMMA 1. *Suppose that T and V are commuting bounded linear operators on the Banach space X. If $T - V$ is compact and T is onto, then V has finite descent.*

PROOF. For each nonnegative integer k, the range, $R(V^k)$, has finite codimension [**1**, Corollary (1.3.7)(b), p. 9] and the map induced by T on $X/R(V^k)$ is onto. Therefore this induced map is one-to-one, so that the null-space $N(T) \subseteq R(V^k)$. Since T is onto, there is a positive number γ for which $\|Tx\| \geqslant \gamma \operatorname{dist}(x, N(T))$ for all x in X. Suppose that x belongs to X and z belongs to $R(V^k)$; then $T(R(V^k)) = R(V^kT) = R(V^k)$ so there is a y in $R(V^k)$ with $Ty = z$. Thus we have $\|Tx - z\| = \|T(x - y)\| \geqslant \gamma \operatorname{dist}(x - y, N(T)) \geqslant \gamma \operatorname{dist}(x, R(V^k))$, since $N(T) \subseteq R(V^k)$. Since this holds for all z in $R(V^k)$, we obtain

$$\operatorname{dist}\big(Tx, R(V^k)\big) \geqslant \gamma \operatorname{dist}\big(x, R(V^k)\big).$$

Suppose V had infinite descent. Then there would be a bounded sequence $\{x_n\}$ with $x_n \in R(V^n)$ and $\operatorname{dist}(x_n, R(V^{n+1})) \geqslant 1$. Let $K = T - V$ and suppose $m > n$. Then $Kx_m - Kx_n = (Kx_m + (T - K)x_n) - Tx_n$. So that

$$\|Kx_m - Kx_n\| \geqslant \operatorname{dist}\big(Tx_n, R(V^{n+1})\big) \geqslant \gamma \operatorname{dist}\big(x_n, R(V^{n+1})\big) \geqslant \gamma.$$

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But this contradicts the compactness of K, so V must have finite descent.

THEOREM 2. *Suppose that T and V are commuting bounded linear operators on the Banach space X with $T - V$ compact.*

(A) *If T is upper semi-Fredholm, then V has finite ascent if and only if T has finite ascent.*

(B) *If T is lower semi-Fredholm, then V has finite descent if and only if T has finite descent.*

PROOF. Suppose first that T is lower semi-Fredholm. Since V is also lower semi-Fredholm [**1**, Corollary (1.3.7)(b), p. 9], it will be enough to show that V has finite descent if T has. Let p be an integer with $R(T^p) = R(T^{p+1})$. Then $R(T^p)$ is a closed subspace of finite codimension [**1**, Corollary (1.3.3), p. 9] and the restriction of T to $R(T^p)$ is onto. Therefore, by Lemma 1, the restriction of V to $R(T^p)$ has finite descent, so that there is an integer k for which

$$R(V^m) \supseteq R(V^m T^p) = R(V^k T^p)$$

for all $m \geqslant k$. Since $R(V^k T^p)$ has finite codimension [**1**, Corollary (1.3.3), p. 9], V has finite descent. This proves (B).

Now suppose that T and V are upper semi-Fredholm. Then T^* and V^* are lower semi-Fredholm, and the ascent of T and V, respectively, equals the descent of T^* and of V^*, respectively [**1**, pp. 7–8]. Part (A) now follows directly from Part (B).

Instead of proving Part (A) from Part (B) in Theorem 2 by duality arguments, we could have proved the dual result to Lemma 1 for T bounded below and then used this result to prove Part (A). The direct proofs are very similar to our proofs of Lemma 1 and Theorem 2(B).

In subsequent papers we will use the results of the present paper to study compact perturbations of more general classes of operators.

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DEPARTMENT OF MATHEMATICS, POMONA COLLEGE, CLAREMONT, CALIFORNIA 91711