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PSEUDO-SIMILARITY FOR MATRICES OVER A FIELD

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ABSTRACT. We call two square matrices A and B (over a ring) pseudo-similar, if matrices X, X^-, $X^{=}$ exist, such that $X^-AX = B$, $XBX^{=}A$, $XX^-X = X$ and $XX^{=}X = X$. We show that if A and B have the same dimension and if the ring is a field, then pseudo-similarity implies similarity, and hence that pseudo-similarity is an equivalence relation.

1. Introduction and notation. In this note we shall investigate the algebraic properties of pseudo-similarity which we define as follows.

DEFINITION. Let $\mathfrak{R}_{m\times n}$ denote the $m \times n$ matrices over a ring with unity. If $A \in \mathfrak{R}_{m\times m}$ and $B \in \mathfrak{R}_{n\times n}$, we say that A is pseudo-similar to B, via X and we write $A \approx B$, if there exists $X \in \mathfrak{R}_{m\times n}$ and two possibly distinct X^-, $X^{=} \in \mathfrak{R}_{n\times m}$ such that

$$\left.\begin{aligned} &(1) \quad X^-AX = B,\\ &(2) \quad XBX^{=} = A,\\ &(3) \quad XX^-X = X,\\ &(4) \quad XX^{=}X = X. \end{aligned}\right\} \qquad (\alpha)$$

In general pseudo-similarity does not imply similarity as seen from the following example of 1×1 matrices over the ring $\mathfrak{L}(\mathbf{R}^\infty)$, of linear transformations on the vector space of real sequences.

Let X be the *right* shift and let Y be the *left* shift. Suppose furthermore that $A = XY$ and $B = I$. Then $XYX = X$, $YAX = B$ and $XBY = A$, so that $A \approx B \not\approx A$.

The theorem in the abstract is established by use of the core-nilpotent decomposition of a square matrix over a field. First we give algebraic properties of pseudo-similarity in a general setting.

As always, any solution to $XMX = X$ will be denoted by X^-, $X^{=}$ etc., while X^+ denotes any solution to $XMX = X$ and $MXM = M$. These generalized inverses are sometimes called "inner" and "reflexive" inverses of X, respectively [**1**]. The set of all inner (reflexive) inverses of $(\cdot)$ will be indicated by $\{(\cdot)^-\}$, $\{(\cdot)^+\}$. Ranges, row spaces and nullspaces of $(\cdot)$ will be denoted by $R(\cdot)$, $RS(\cdot)$ and $N(\cdot)$, respectively.

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2. Properties of $\approx$.

THEOREM 1. *Suppose $A \approx B$ with $X, X^-, X^=$ as in the definition (α). Then the following are valid.*

(1) $A = XX^-AXX^= = AXX^= = XX^-A$, $B = X^-XBX^=X = X^-XB = BX^=X$.

(2) $X^-A^kX = B^k$, $XB^kX^= = A^k$, $k = 1, 2, \ldots$.

(3) $\{A^-X\} \subseteq \{(X^-A)^-\}$, $\{XB^-\} \subseteq \{(BX^=)^-\}$, $\{A^+X\} \subseteq \{(X^-A)^+\}$, $\{XB^+\} \subseteq \{(BX^=)^+\}$.

(4) (i) $\{X^=A^-X\} \subseteq \{B^-\}$, (ii) $\{XB^-X^-\} \subseteq \{A^-\}$, (iii) $\{X^=A^+X\} \subseteq \{B^+\}$, (iv) $\{XB^+X^-\} \subseteq \{A^+\}$.

(5) (i) $\{X^=XB^-X^-X\} \subseteq \{B^-\}$, (ii) $\{XX^=A^-XX^-\} \subseteq \{A^-\}$, (iii) $\{X^=XB^+X^-X\} \subseteq \{B^+\}$, (iv) $\{XX^=A^+XX^-\} \subseteq \{A^+\}$.

(6) $\{X^-AA^-X\} \subseteq \{(BB^-)^+\}$, $\{X^=A^-AX\} \subseteq \{(B^-B)^+\}$, $\{XBB^-X^-\} \subseteq \{(AA^-)^+\}$, $\{XB^-BX^=\} \subseteq \{(A^-A)^+\}$.

(7) (i) $B^-B = I \Rightarrow X^-X = I \Leftrightarrow X^=X = I \Leftrightarrow BB^- = I$, (ii) $A^-A = I \Rightarrow XX^= = I \Leftrightarrow XX^- = I \Leftrightarrow AA^- = I$.

(8) $R(A) \cong R(B)$, $RS(A) \cong RS(B)$ *as $\mathfrak{R}$-modules.*

In these expressions X^- and $X^=$ are the inner inverses of X that occur in (1)–(4), while A^-, B^- are any inner inverses of A, B.

PROOF. (1) Substitution of $X^-AX = B$ in $XBX^= = A$ yields $A = XX^-AXX^=$, which on successive premultiplication by XX^- and postmultiplication by $XX^=$ shows that $A = XX^-A = AXX^=$. Similarly $B = X^-XBX^=X = X^-XB = BX^=X$.

(2) Clearly $B^2 = X^-A(XX^-A)X = X^-A^2X$ and $A^2 = X(BX^=X)BX^= = XB^2X^=$. The proof now follows by induction.

(3) Obviously, $X^-A(A^-X)X^-A = X^-AA^-(XX^-A) = X^-AA^-A = X^-A$ and $A^+X(X^-A)A^+X = A^+(XX^-A)A^+X = A^+AA^+X = A^+X$.

The remaining results follow by symmetry.

(4) We verify that $B(X^=A^-X)B = (BX^=)A^-(XB) = X^-AA^-AX = X^-AX = B$ and $(X^=A^+X)B(X^=A^+X) = X^=A^+(XBX^=)A^+X = X^=A^+AA^+X = X^=A^+X$. The remaining parts follow by symmetry.

(5) These follow from (4) on substituting 4(ii) in 4(i) and vice-versa. Similarly for 4(iii) and 4(iv).

(6) First note that $BB^-(X^-AA^-X)BB^- = BB^-X^-AA^-(XB)B^- = BB^-X^-AA^-AXB^- = BB^-BB^- = BB^-$. Secondly,

$$(X^-AA^-X)BB^-(X^-AA^-X) = X^-AA^-XBB^-(X^-A)A^-X$$
$$= X^-AA^-XBB^-BX^=A^-X = X^-AA^-AA^-X = X^-AA^-X.$$

One can establish the other results in an analogous way.

(7) From (6), $BB^- = I \Rightarrow X^-AA^-X = I$ and hence $X^-X = X^-AA^-XX^-X = X^-AA^-X = I$. Similarly for the other results.

(8) Define φ: $R(A) \to R(B)$ by $\varphi(A\mathbf{s}) = BX^=\mathbf{s}$, $\mathbf{s} \in \mathfrak{R}^n$. Then φ is clearly a well-defined $\mathfrak{R}$-homomorphism. If $BX^=\mathbf{s}_1 = BX^=\mathbf{s}_2$ then $A\mathbf{s}_1 = XBX^=\mathbf{s}_1 = XBX^=\mathbf{s}_2 = A\mathbf{s}_2$, so that φ is one-one.

Lastly, $By = BX^{=}Xy = \varphi(AXy)$, implying φ is also onto. Thus $R(A) \cong R(B)$. In the same way, if we define $\theta(\mathbf{u}^T A) = \mathbf{u}^T XB$, it then follows that $RS(A) \cong RS(B)$. The proof of the theorem is now complete.

Let us now rephrase the condition of pseudo-similarity in a form which is more transparent when viewed from the theory of matrix equations.

COROLLARY 1. *The following are equivalent.*
(i) $A \approx B$ *via* X.
(ii) $AX = XB, AXX^{=} = A, X^{-}XB = B, XX^{-}X = XX^{=}X = X$.
(iii) $BX^{=} = X^{-}A, XX^{-}A = A, BX^{=}X = B, XX^{-}X = XX^{=}X = X$.

PROOF. (i) $\Leftrightarrow$ (ii). Let $A \approx B$. By Theorem 1, part 1, $A = AXX^{=} = XX^{-}A$, $B = X^{-}XB = BX^{=}X$. Hence $X^{-}AX = B \Rightarrow XB = XX^{-}AX = AX$. Conversely substituting $AX = XB$ in $A = AXX^{=}$ and $B = X^{-}XB$ yields $A = XBX^{=}$ and $B = X^{-}AX$ respectively. The equivalence of (i) and (iii) follows similarly.

REMARKS.

(1) One can use these conditions to verify the following range and row-space equations: $R(A) = R(XB)$, $R(B) = R(X^{-}A)$, $RS(A) = RS(BX^{=})$, $RS(B) = RS(AX)$.

(2) The equivalence between (ii) and (iii) also holds *without* X^{-}, $X^{=}$ being inner inverses.

3. Main Theorem. Let us now specialize to the case where $\mathfrak{R}$ is a field $\mathfrak{F}$, and suppose that A, B and X have the same dimension n. In this case the product rule for determinants shows that the main assertion (Theorem 2) is obvious if any of A, B or X is invertible.

Hence without loss of generality we may assume all three matrices to be singular.

The following additional corollaries of Theorem 1 are then immediate.

(1) If $p(\lambda) = p_0 + p_1\lambda + \cdots + p_n\lambda^n$ and $p(A) = 0$, then the minimal polynomial ψ_A divides $p(\lambda)$ and hence $p_0 = 0$.

Thus by part 2 of Theorem 1,

$$0 = \sum_1^n p_i A^i \Rightarrow 0 = \sum_1^n p_i X^{-} A^i X = \sum_1^n p_i B^i = p(B).$$

The converse follows by symmetry. In particular, the minimal polynomials ψ_A and ψ_B are equal, so that A and B have equal indices, which are defined by $\operatorname{index}(A) = \min_k \{R(A^k) = R(A^{k+1})\}$ [**1**, p. 170]. Also, if Δ_M denotes the characteristic polynomial of M, then $\Delta_B = \Delta_{X^{-}AX} = \Delta_{XX^{-}A} = \Delta_A$, since $\Delta_{PQ} = \Delta_{QP}$ always holds for square matrices P and Q over a commutative ring.

(2) The ranks of A^k and B^k must be equal for all k = 0, 1, . . . , as seen from part 2. This has far reaching consequences. Indeed, this implies that A and B have the same nilpotent Jordan blocks $J_i(0)$, because the zero Weyr characteristics are the same [**2**]. This may also be seen from Fitting's core-nilpotent decomposition $P^{-1}AP = \operatorname{diag}[U_A, N_A]$, where U_A is invertible and

N_A contains all the nilpotent Jordan blocks (even over nonclosed fields). Thus $A(I - AA^d) \approx B(I - BB^d)$, where $(\cdot)^d$ is the Drazin inverse of A, defined by $P^{-1}A^dP = \operatorname{diag}[U_A^{-1}, 0]$ [1, p. 169].

THEOREM 2. *Let $A, B \in \mathfrak{F}_{n\times n}$. Then $A \simeq B \Leftrightarrow A \approx B$.*

PROOF. The sufficiency is clear. Suppose therefore that $X^-AX = B$ and $XBX^= = A$ for some $X^-, X^= \in \mathfrak{F}_{n\times n}$, so that the conditions of Corollary 1, part (ii), hold. As observed earlier, we may without loss of generality assume that A, B and X are singular. Now, consider the core-nilpotent decomposition: [1, p. 175],

$$P^{-1}AP = J_A = \operatorname{diag}[U_A, N_A], \qquad Q^{-1}BQ = J_B = \operatorname{diag}[U_B, N_B].$$

Then $AX = XB$, $AXX^= = A$, $X^-XB = B$ imply $J_AY = YJ_B$, $J_AYY^= = J_A$, $Y^-YJ_B = J_B$ where $Y = P^{-1}XQ$, $Y^- = Q^{-1}X^-P$ and $Y^= = Q^{-1}X^=P$.

Since $A \simeq B$, $\Delta_A = \Delta_B$ as noted above and it follows that U_A and U_B have the same size. In addition, Theorem 1, part 2, ensures that $N_A = N_B$. We now proceed by partitioning Y conformally as

$$Y = \begin{bmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{bmatrix}$$

and substitute in $J_AY = YJ_B$. This yields

$$U_AY_3 = Y_3N_B, \quad N_AY_2 = Y_2U_B, \quad U_AY_1 = Y_1U_B, \quad N_AY_4 = Y_4N_B. \tag{3.1}$$

Since there are no common elementary divisors for U_A and N_B it follows that [3] $Y_3 = 0$. Similarly $Y_2 = 0$. Suppose we now partition Y^- as

$$Y^- = \begin{bmatrix} S & * \\ * & * \end{bmatrix},$$

and hence

$$Y^-Y = \begin{bmatrix} SY_1 & * \\ * & * \end{bmatrix}.$$

So, $Y^-YJ_B = J_B$ shows in particular that $SY_1U_B = U_B$. Since U_B is invertible we may deduce that Y_1 is also invertible. Returning to (3.1) we then have $Y_1^{-1}U_AY_1 = U_B$ and hence that

$$\begin{bmatrix} Y_1^{-1} & 0 \\ 0 & I \end{bmatrix} J_A \begin{bmatrix} Y_1 & 0 \\ 0 & I \end{bmatrix} = J_B,$$

or equivalently, $A \simeq B$, as desired.

If A and B are square but of *different* dimensions then we may add zeros to the matrices X, X^- etc., to obtain the following generalization of the above result.

COROLLARY 2. *If $A \in \mathfrak{F}_{m\times m}$, $B \in \mathfrak{F}_{n\times n}$ and $m > n$, then $A \simeq B \Leftrightarrow A \approx$* diag $[B, 0]$.

REMARKS. What we have proven is in fact that pseudo-similarity is an equivalence relation. In particular, if $X^-AX = B$, $XBX^= = A$, $Y^-BY = C$ and $YCY^= = B$, then there exist Z, Z^- and $Z^=$ such that $Z^-AZ = C$ and $ZCZ^= = A$. This matrix Z however, *need not* equal XY, nor Z^- equal Y^-X^-, even though

$$(Y^-X^-)A(XY) = C \quad \text{and} \quad (XY)C(Y^=X^=) = A.$$

This relation does hold if Y^-X^- and $Y^=X^=$ are both inner inverses of XY. Equivalent local conditions for this were obtained in [4, Proposition 2.1(3)]. It may be shown, using the general form $\{Y^-\} = Y^- + (I - Y^-Y)H + K(I - YY^-)$, H, K arbitrary, that in fact $\{Y^-\}\{X^-\} \subseteq \{(XY)^-\} \Leftrightarrow N(X) \subseteq R(Y)$, which clearly will not (and cannot) be true in general.

We close with the conjecture that Theorem 2 holds for matrices over a *unit-regular* ring, for which each $a \in R$ has at least one unit (= invertible) inner inverse a^-.

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