PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

A GENERALIZATION OF THE RIESZ-HERGLOTZ THEOREM ON REPRESENTING MEASURES

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ABSTRACT. A simple construction is given that obtains maximal representing measures for positive harmonic functions on a domain W as the weak* limits of finite sums of point masses on $[0, +\infty]^W$. This new standard result, new even for the unit disk, is established for very general elliptic differential equations and domains, in fact, for a Brelot harmonic space, using nonstandard analysis.

Let **C** denote the complex plane, D the unit disk $\{z \in \mathbf{C}: |z| < 1\}$, and $P(z, a)$ the Poisson kernel $(|z|^2 - |a|^2)/|z - a|^2$. For each positive $r \leqslant 1$, let $C_r = \{z \in \mathbf{C}: |z| = r\}$ and let λ_r be $1/2\pi r$ times Lebesgue measure on C_r. Let $\Phi_{x_0}^D$ denote the positive harmonic functions on D with value 1 at the origin x_0. By a theorem first established by F. Riesz [11] but usually attributed to G. Herglotz [3], there is for each $h \in \Phi_{x_0}^D$ a unique representing measure ν_h on C_1; that is,

$$h(a) = \int_{C_1} P(z, a)\, d\nu_h(z)$$

for each $a \in D$. Moreover, ν_h is the weak* limit as $r \to 1$ of the measures $h\lambda_r$ on C_r. Note that ν_h is both a measure on C_1 and on the harmonic functions $\{P(z, \cdot): z \in C_1\}$; these functions are the extreme points in the convex set $\Phi_{x_0}^D$.

For a general open and connected set W in a Euclidean space or Riemann surface, we let $\Phi_{x_0}^W$ denote the positive harmonic functions on W with value 1 at some $x_0 \in W$. Now it is the Martin boundary theory ([9] or [2]) and in general the Choquet theory [10] (both of which, historically, have roots in the Riesz-Herglotz Theorem) that give for each $h \in \Phi_{x_0}^W$ a unique representing measure ν_h (maximal with respect to the Choquet ordering) on the extreme elements of $\Phi_{x_0}^W$. The general Choquet theory does not give a simple construction of ν_h. Martin's theory does allow a repetition for W of the construction of ν_h for the disk, but one must imbed W in its Martin compactification $\overline{W}^M$. For many simple domains such as the Lebesgue spine,

Received by the editors July 18, 1977.

AMS (MOS) subject classifications (1970). Primary 31A10, 31B10, 31D05; Secondary 02H20.

Key words and phrases. Herglotz Theorem, representing measures, Brelot harmonic spaces, nonstandard analysis, weak* convergence.

[1]This work was supported by a grant from the U. S. National Science Foundation.

the structure of $\overline{W}^M$ and, therefore, of $C(\overline{W}^M)$ is still unclear. It was only in 1970, for example, that a deep and difficult paper by Hunt and Wheeden [4] established the equivalence between Martin's boundary and the Euclidean boundary for a Lipschitz domain.

In this note, we give a simple construction of representing measures that is new even for the unit disk and is valid for very general elliptic differential equations and domains. For the disk D and $h \in \Phi_{x_0}^D$, this construction is obtained by modifying the classical construction as follows:

For each positive $r < 1$ and each $x \in D$ with $|x| < r$, let μ_x^r denote harmonic measure on C_r with respect to x. That is, $\mu_x^r = P(\cdot, x)\lambda_r$ and for the origin x_0, $\mu_{x_0}^r = \lambda_r$. Let $\mathcal{P}_r$ be a partition of C_r into a finite number of intervals A_i with points y_i chosen from each A_i, and let $\delta_{y_i}^r$ be unit mass at y_i for each i. A trivial modification of the classical construction of ν_h as the weak* limit of the measures $h\lambda_r$ shows that the measures

$$\sum_i h(y_i)\, \mu_{x_0}^r(A_i)\delta_{y_i}^r$$

converge to ν_h in the weak* topology as $r \to 1$ and the intervals in $\mathcal{P}_r$ become smaller. For our new construction, we replace each measure $\delta_{y_i}^r$ with unit mass δ_i^r on the harmonic function $\mu_x^r(A_i)/\mu_{x_0}^r(A_i)$ extended by 0 on and outside C_r. The measures

$$\sum_i h(y_i)\, \mu_{x_0}^r(A_i)\delta_i^r$$

converge to ν_h in the weak* topology for regular Borel measures on $[0, +\infty]^D$ as $r \to 1$ and the intervals in $\mathcal{P}_r$ become smaller. Note that by the Stone-Weierstrass Theorem, the set of continuous real-valued functions on $[0, +\infty]^D$ is the uniform closure of the algebra generated by constants and all finite truncations of evaluations at points of D.

Our result is stated for a Brelot harmonic space $(\mathcal{H}, W)$ with **1** superharmonic on W. (See [2] or [5].) We do not need to add to Brelot's axioms the additional assumptions necessary for the Martin compactification theory. The reader who is interested in only a special case of the Brelot theory, e.g., solutions of Laplace's equation or more general elliptic differential equations in R^n, can read the rest of this note with the assumption that the case of interest is the one being discussed.

Fix $x_0 \in W$. For each region (i.e., connected open set) $\Omega \subset W$, let $\mathcal{H}_\Omega = \{h \in \mathcal{H}\colon \text{domain } h = \Omega\}$. If $x_0 \in \Omega$, let $\Phi_{x_0}^\Omega = \{h \in \mathcal{H}_\Omega\colon h > 0 \text{ and } h(x_0) = 1\}$. We assume that $\Phi_{x_0}^W \neq \varnothing$. Let $Y = [0, +\infty]^W$ with the product topology. The restriction of the product topology to $\Phi_{x_0}^W$ is the topology of uniform convergence on compact sets, and $\Phi_{x_0}^W$ is a compact metric space in this topology. (See [6] and [8].) Let E_{x_0} be the set of extreme points in the compact convex set $\Phi_{x_0}^W$; E_{x_0} is a G_δ in $\Phi_{x_0}^W$ and therefore is a Borel set in Y. By Choquet's Theorem, for each $h \in \Phi_{x_0}^W$, there is a unique regular Borel measure ν_h supported by E_{x_0} such that for each $x \in W$

$$h(x) = \int_{E_{x_0}} T_x(g)\, d\nu_h(g).$$

Here T_x denotes evaluation at x. (See [6].)

Given a region Ω in W, we let $\bar{\Omega}$ denote its closure and $C(\partial\Omega)$, the continuous real-valued functions on $\partial\Omega = \bar{\Omega} - \Omega$. We call Ω a regular inner region if $\bar{\Omega}$ is compact and each $f \in C(\partial\Omega)$ has a unique continuous extension h_f on $\bar{\Omega}$ such that $h_f|\Omega \in \mathcal{H}_\Omega$ and $h_f \geqslant 0$ if $f \geqslant 0$. Since the mapping $f \to h_f$ is positive and linear, for each $x \in \Omega$ there is a Radon measure μ_x^Ω, called harmonic measure for x and Ω, on $\partial\Omega$ such that for each $f \in C(\partial\Omega)$,

$$h_f(x) = \int_{\partial\Omega} f\, d\mu_x^\Omega.$$

If $h \in \mathcal{H}_\Omega$, $h \geqslant 0$, we extend h with the value 0 on $W - \Omega$ so that $h \in Y = [0, +\infty]^W$.

Let $\mathcal{I}$ consist of sets of the form $\alpha = \{\varepsilon, K\}$, where ε is a positive real number, and K is a compact set in W with $x_0 \in K$. Given $\alpha = \{\varepsilon_\alpha, K_\alpha\}$ and $\beta = \{\varepsilon_\beta, K_\beta\}$ in $\mathcal{I}$, we say that $\alpha \leqslant \beta$ if $\varepsilon_\alpha \geqslant \varepsilon_\beta$, and $K_\alpha \subset K_\beta$. Clearly, $(\mathcal{I}, \leqslant)$ is a directed set.

Now fix $h \in \Phi_{x_0}^W$. Given α in $\mathcal{I}$, associate with α and h a measure ν_h^α as follows:

Choose a regular inner region $\Omega \supset K_\alpha$ [5, Theorem 4.3]. By the equicontinuity of $\Phi_{x_0}^W$ on $\partial\Omega$ [8], we may partition $\partial\Omega$ into a finite number of measurable sets A_i, $1 \leqslant i \leqslant m$, so that for each function $f \in \Phi_{x_0}^W$,

$$\sup_{A_i} f - \inf_{A_i} f < \varepsilon_\alpha.$$

By discarding null sets, we may assume that $\mu_{x_0}^\Omega(A_i) > 0$ for each i. Choose $y_i \in A_i$, $1 \leqslant i \leqslant m$. Let δ_i^Ω be the unit mass on the harmonic function

$$\frac{\mu_x^\Omega(A_i)}{\mu_{x_0}^\Omega(A_i)} \in \Phi_{x_0}^\Omega, \quad x \in \Omega.$$

Then δ_i^Ω is a regular Borel measure on Y. Let ν_h^α be the sum

$$\sum_{i=1}^{m} h(y_i)\, \mu_{x_0}^\Omega(A_i)\delta_i^\Omega.$$

THEOREM. *The regular Borel measures ν_h^α converge with respect to the ordering $\leqslant$ on $\mathcal{I}$ to the representing measure ν_h supported by E_{x_0} in the weak* topology for regular Borel measures on Y. That is, for each continuous real-valued function F on Y,*

$$\lim_\alpha \int_Y F(y)\, d\nu_h^\alpha(y) = \int_{E_{x_0}} F(y)\, d\nu_h(y).$$

PROOF. The proof is nonstandard. It is a modification of the proof of Theorem 4.6 of [6]. Here, however, one uses arbitrary continuous functions F instead of just point evaluations T_x. One may associate the domain Ω and partition $\{A_i\}$ in the proof of Theorem 4.6 of [**6**] with an arbitrary infinite β in $^*\mathcal{I}$; that is $\beta > {}^*\alpha$ for each standard $\alpha \in \mathcal{I}$. Since for each infinite $\beta \in {}^*\mathcal{I}$

$$\int_{^*Y} {}^*F(y)\, d\nu_h^\beta(y) \simeq \int_{E_{x_0}} F(y)\, d\nu_h(y),$$

the theorem follows from A. Robinson's criterion for convergence of nets [**12**, Theorems 4.2.4 and 4.2.5].

Note. One can use the existence of a countable exhaustion of W by compact sets K_n (see, for example, [**6**, p. 161]) to obtain a cofinal sequence $\nu_h^{\{K_n, 1/n\}}$ in the net ν_h^α.

For results similar to the above, see [**6**, Theorems 4.8 and 4.9]. For other results using this nonstandard method of dealing with standard weak* convergence arguments in potential theory, the reader is referred to [**6**, Theorems 3.1 and 6.5]. Weak* convergence for general spaces is discussed by Anderson and Rashid in [**1**]; weak* cluster points and further examples are discussed by the author in [**7**].

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