PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

CAPACITY AND EQUIDISTRIBUTION FOR HOLOMORPHIC MAPS FROM $\mathbf{C}^2$ TO $\mathbf{C}^2$

ROBERT E. MOLZON

ABSTRACT. The relationship between equidistribution for holomorphic maps and sets of capacity zero are investigated.

0. Introduction. For nondegenerate holomorphic mappings $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ the Fatou-Bieberbach example gives a map in which the image $f(\mathbf{C}^2)$ omits an open set. In [1] Chern-Wu prove the following:

THEOREM. *Let* $f\colon \mathbf{C}^2 \to \mathbf{P}^2$ *be a holomorphic mapping and let* $T_1'(r) = dT_1(r)/d\log r$. *If* $\lim_{r\to\infty}[T_1'(r)/T_2(r)] = 0$ *then the image* $f(\mathbf{C}^2)$ *is dense in* $\mathbf{P}^2$. *(*$T_1(r)$ *and* $T_2(r)$ *are the order functions in the Nevanlinna theory.)*

Suppose $f\colon \mathbf{C}^2 \to \mathbf{C}^2$ is a holomorphic map. In this paper we prove the

THEOREM. *If* $\lim_{r\to\infty}[T''(r)/T_2(r)] = 0$ *and* $\lim_{r\to\infty}[\log r/T_2(r)] = 0$ *then* f *takes on every value* $w \in \mathbf{C}^2$ *infinitely often except possibly for a set* w *of* $2+\varepsilon$ *capacity zero. Here* $T''(r) = rT_1'(r)$, *with* $T_1'(r) = dT_1(r)/dr$ *and capacity refers to Newtonian capacity.*

1. Notation. Let $\tau = \log|\zeta|^2$ be the standard exhaustion function on $\mathbf{C}^2$. If $W \in \mathbf{P}^2$ is the intersection of perpendicular lines A and B let $\Lambda_W = \log[|Z|^2/(|\langle Z, A\rangle|^2 + |\langle Z, B\rangle|^2)](\omega + \omega_0)$ where $\omega_0 = dd^c \log[|\langle Z, A\rangle|^2 + |\langle Z, B\rangle|^2]$ and $\omega = dd^c \log|Z|^2$ are defined on $\mathbf{P}^2$. Let $n(W, t) = \operatorname{card}(f^{-1}(W) \cap \{|\zeta| \leqslant t\})$ and $N(W, r) = \int_0^r n(W, t)\, d\log t$. We also have the two order functions, the proximity term and the error term:

$$T_1(r) = \int_0^r \left(\int_{|\zeta|<t} f^*\omega \wedge dd^c\tau\right) d\log t; \quad T_2(r) = \int_0^r \left(\int_{|\zeta|<t} f^*\omega \wedge f^*\omega\right) d\log t,$$

$$m(W, r) = \int_{|\zeta|=r} f^*\Lambda_W \wedge d^c\tau; \quad S(W, r) = \int_{|\zeta|<r} f^*\Lambda_W \wedge dd^c\tau.$$

The First Main Theorem of Nevanlinna then states:

$$N(W, r) + m(W, r) = T_2(r) + S(W, r) + C.$$

(We assume in the entire discussion that $f^{-1}(W) \neq 0$.)

Received by the editors November 20, 1977.
AMS (MOS) subject classifications (1970). Primary 32H25; Secondary 31B15.

2. Capacity. If $E \subset \mathbf{C}^2$ and μ is a measure supported on E then the α potential of μ is

$$V_\mu(x) = \int_{\mathbf{C}^2} \frac{1}{|w - x|^\alpha} \, d\mu(w).$$

Write $K_\alpha(x) = 1/|x|^\alpha$. The energy of μ is $\mathcal{E}(\mu) = \int_{\mathbf{C}} V_\mu(x)\, d\mu(x)$. Given a Borel set E and a real number $Q > 0$ there is a unique measure μ such that $\mu(\mathbf{C}^2) = \mu(E) = Q$ and $\mathcal{E}(\mu)$ is minimal. Such a measure is called an equilibrium measure. Let E be a Borel set in $\mathbf{C}^2$ and μ the equilibrium measure supported on E. Let $V = \max V_\mu(x)$. If V exists the α capacity of E is $C_\alpha(E) = Q/V$. If V does not exist then E is of α capacity zero.

3. Proof of Theorem. We suppose $f\colon \mathbf{C}^2 \to \mathbf{C}^2$. Regarding $\mathbf{C}^2 \subset \mathbf{P}^2$ we have the FMT as in §1. We need the following lemma concerning the proximity term:

LEMMA. *Let $E \subset \mathbf{C}^2$ be a set of positive $2 + \varepsilon$ capacity. Then there is a measure μ supported on E such that $\int_{\mathbf{C}^2} m(r, w)d\mu(w) \leqslant rT_1'(r)$.*

PROOF. Let μ be the equilibrium measure on E. The $2 + \varepsilon$ potential is $V_\mu(x) = \int_E 1/(|x - w|^{2+\varepsilon})d\mu(w)$. Normalize μ such that $\mu(E) = 1$. Let $V = \max V_\mu(x)$. Since E has positive capacity $V < \infty$. Now

$$\int_{\mathbf{C}^2} m(r, w)d\mu(w) = \int_E m(r, w)d\mu(w) = \int_E \left(\int_{|\zeta| = r} f^*\Lambda_w \wedge d^c\tau \right) d\mu(w)$$
$$= \int_{|\zeta| = r} f^*\left(\int_E \Lambda_w d\mu(w) \right) \wedge d^c\tau$$

where Λ_w is the Λ_W of §1 in local coordinates, that is, regarding $\mathbf{C}^2 \subset \mathbf{P}^2$. So

$$\Lambda_0(x) = \log\left[\left(1 + |x_1|^2 + |x_2|^2\right) / \left(|x_1|^2 + |x_2|^2\right) \right](\omega + \omega_0)$$

where $\omega = dd^c \log(1 + |x_1|^2 + |x_2|^2)$ and $\omega_0 = dd^c \log(|x_1|^2 + |x_2|^2)$. Hence

$$\Lambda_w(x) = \left[1/\left(|w - x|^2\right) \right] \log\left[\left(1 + |w - x|^2\right)/|w - x|^2 \right] \cdot \beta(x, w)$$

where β is a uniformly bounded (1,1) form. Hence $\beta(x, w) \leqslant K \cdot \omega(x)$. Now using the inequality $\log[(1 + x)/x] \leqslant c/x^\varepsilon$ where c depends only upon ε we have $\Lambda_w \leqslant c/|x - w|^{2+\varepsilon} \cdot \omega(x)$. Therefore

$$\int_{|\zeta| = r} f^*\left(\int_E \Lambda_w d\mu(w) \right) \wedge d^c\tau \leqslant c \int_{|\zeta| = r} f^*\left(\left[\int_E K_{2+\varepsilon}(x - w) d\mu(w) \right] \omega \right) \wedge d^c\tau$$
$$\leqslant c \int_{|\zeta| = r} f^*(V_\mu(x)\omega) \wedge d^c\tau \leqslant \int_{|\zeta| = r} f^*\omega \wedge d^c\tau.$$

Since

$$T_1(r) = \int_0^r \left\{ \int_{s=0}^t \left(\int_{|\zeta| = s} f^*\omega \wedge d^c\tau \right) ds \right\} d \log t$$

we finally obtain $\int_{|\zeta|=r} f^*(\int_E \Lambda_w d\mu(w)) \wedge d^c\tau \leqslant rT_1'(r)$, where $T_1'(r) = dT_1(r)/dr$. □

PROOF OF THEOREM. The FMT states $N(w, r) + m(w, r) = T_2(r) + S(w, r) + C$.

(*) Hence $T_2(r) \leqslant N(w, r) + m(w, r) + C$. Suppose there exists a set of positive $2 + \varepsilon$ capacity $E \subset \mathbf{C}^2$ such that f takes on values in E only finitely many times. Let $E_m \subset E$ be the set such that f takes on values in E_m at most m times. Then $E = \bigcup E_m$. Since E has positive $2 + \varepsilon$ capacity one of the E_m, say E_{m_0}, must have positive capacity. Let μ be the equilibrium measure on E_{m_0} normalized such that $\mu(E_{m_0}) = 1$. Integrating (*) with respect to $d\mu(w)$ over $\mathbf{C}^2$ we have:

$$T_2(r) \leqslant \int_C N(w, r) d\mu(w) + \int_C m(w, r) d\mu(w) + C$$

$$= \int_{E_{m_0}} \left(\int_0^r n(w, t) d \log t \right) d\mu(w) + \int_{E_{m_0}} m(w, r) d\mu(w) + C,$$

$$T_2(r) \leqslant m_0 \log r + cT''(r) + C.$$

Since $\lim_{r\to\infty}[T''(r)/T_2(r)] = 0$ and $\lim_{r\to\infty}[\log r/T_2(r)] = 0$ we have a contradiction. Hence no such set E exists. □

BIBLIOGRAPHY

1. J. Carlson, *A moving lemma for the transcendental Bezout problem*, Ann. of Math. (2) **103** (1976), 305–330.

2. W. H. J. Fuchs, *Topics in the theory of functions of one complex variable*, Van Nostrand, Princeton, N. J., 1967.

3. P. Griffiths, *On the Bezout problem for entire analytic sets*, Ann. of Math. (2) **100** (1974), 533–552.

4. H. Wu, *Remarks on the First Main Theorem of equidistribution theory*. I, II, III, J. Differential Geometry **2** (1968), 197–202; **3** (1969), 83–94, 369–384.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506