

CAPACITY AND EQUIDISTRIBUTION FOR HOLOMORPHIC MAPS FROM \mathbb{C}^2 TO \mathbb{C}^2

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ABSTRACT. The relationship between equidistribution for holomorphic maps and sets of capacity zero are investigated.

0. Introduction. For nondegenerate holomorphic mappings $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ the Fatou-Bieberbach example gives a map in which the image $f(\mathbb{C}^2)$ omits an open set. In [1] Chern-Wu prove the following:

THEOREM. *Let $f: \mathbb{C}^2 \rightarrow \mathbb{P}^2$ be a holomorphic mapping and let $T_1'(r) = dT_1(r)/d \log r$. If $\lim_{r \rightarrow \infty} [T_1'(r)/T_2(r)] = 0$ then the image $f(\mathbb{C}^2)$ is dense in \mathbb{P}^2 . ($T_1(r)$ and $T_2(r)$ are the order functions in the Nevanlinna theory.)*

Suppose $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a holomorphic map. In this paper we prove the

THEOREM. *If $\lim_{r \rightarrow \infty} [T''(r)/T_2(r)] = 0$ and $\lim_{r \rightarrow \infty} [\log r/T_2(r)] = 0$ then f takes on every value $w \in \mathbb{C}^2$ infinitely often except possibly for a set w of $2 + \epsilon$ capacity zero. Here $T''(r) = rT_1'(r)$, with $T_1'(r) = dT_1(r)/dr$ and capacity refers to Newtonian capacity.*

1. Notation. Let $\tau = \log |\zeta|^2$ be the standard exhaustion function on \mathbb{C}^2 . If $W \in \mathbb{P}^2$ is the intersection of perpendicular lines A and B let $\Lambda_W = \log[|Z|^2/(|\langle Z, A \rangle|^2 + |\langle Z, B \rangle|^2)](\omega + \omega_0)$ where $\omega_0 = dd^c \log[|\langle Z, A \rangle|^2 + |\langle Z, B \rangle|^2]$ and $\omega = dd^c \log|Z|^2$ are defined on \mathbb{P}^2 . Let $n(W, t) = \text{card}(f^{-1}(W) \cap \{|\zeta| \leq t\})$ and $N(W, r) = \int_0^r n(W, t) d \log t$. We also have the two order functions, the proximity term and the error term:

$$T_1(r) = \int_0^r \left(\int_{|\zeta| < t} f^* \omega \wedge dd^c \tau \right) d \log t; \quad T_2(r) = \int_0^r \left(\int_{|\zeta| < t} f^* \omega \wedge f^* \omega \right) d \log t,$$

$$m(W, r) = \int_{|\zeta|=r} f^* \Lambda_W \wedge d^c \tau; \quad S(W, r) = \int_{|\zeta| < r} f^* \Lambda_W \wedge dd^c \tau.$$

The First Main Theorem of Nevanlinna then states:

$$N(W, r) + m(W, r) = T_2(r) + S(W, r) + C.$$

(We assume in the entire discussion that $f^{-1}(W) \neq \emptyset$.)

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2. Capacity. If $E \subset \mathbb{C}^2$ and μ is a measure supported on E then the α potential of μ is

$$V_\mu(x) = \int_{\mathbb{C}^2} \frac{1}{|w - x|^\alpha} d\mu(w).$$

Write $K_\alpha(x) = 1/|x|^\alpha$. The energy of μ is $\mathcal{E}(\mu) = \int_{\mathbb{C}^2} V_\mu(x) d\mu(x)$. Given a Borel set E and a real number $Q > 0$ there is a unique measure μ such that $\mu(\mathbb{C}^2) = \mu(E) = Q$ and $\mathcal{E}(\mu)$ is minimal. Such a measure is called an equilibrium measure. Let E be a Borel set in \mathbb{C}^2 and μ the equilibrium measure supported on E . Let $V = \max V_\mu(x)$. If V exists the α capacity of E is $C_\alpha(E) = Q/V$. If V does not exist then E is of α capacity zero.

3. Proof of Theorem. We suppose $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Regarding $\mathbb{C}^2 \subset \mathbb{P}^2$ we have the FMT as in §1. We need the following lemma concerning the proximity term:

LEMMA. *Let $E \subset \mathbb{C}^2$ be a set of positive $2 + \epsilon$ capacity. Then there is a measure μ supported on E such that $\int_{\mathbb{C}^2} m(r, w) d\mu(w) \leq rT'_1(r)$.*

PROOF. Let μ be the equilibrium measure on E . The $2 + \epsilon$ potential is $V_\mu(x) = \int_E 1/(|x - w|^{2+\epsilon}) d\mu(w)$. Normalize μ such that $\mu(E) = 1$. Let $V = \max V_\mu(x)$. Since E has positive capacity $V < \infty$. Now

$$\begin{aligned} \int_{\mathbb{C}^2} m(r, w) d\mu(w) &= \int_E m(r, w) d\mu(w) = \int_E \left(\int_{|\zeta|=r} f^* \Lambda_w \wedge d^c \tau \right) d\mu(w) \\ &= \int_{|\zeta|=r} f^* \left(\int_E \Lambda_w d\mu(w) \right) \wedge d^c \tau \end{aligned}$$

where Λ_w is the Λ_w of §1 in local coordinates, that is, regarding $\mathbb{C}^2 \subset \mathbb{P}^2$. So

$$\Lambda_0(x) = \log \left[(1 + |x_1|^2 + |x_2|^2) / (|x_1|^2 + |x_2|^2) \right] (\omega + \omega_0)$$

where $\omega = dd^c \log(1 + |x_1|^2 + |x_2|^2)$ and $\omega_0 = dd^c \log(|x_1|^2 + |x_2|^2)$. Hence

$$\Lambda_w(x) = \left[1 / (|w - x|^2) \right] \log \left[(1 + |w - x|^2) / |w - x|^2 \right] \cdot \beta(x, w)$$

where β is a uniformly bounded (1,1) form. Hence $\beta(x, w) \leq K \cdot \omega(x)$. Now using the inequality $\log[(1 + x)/x] \leq c/x^\epsilon$ where c depends only upon ϵ we have $\Lambda_w \leq c/|x - w|^{2+\epsilon} \cdot \omega(x)$. Therefore

$$\begin{aligned} \int_{|\zeta|=r} f^* \left(\int_E \Lambda_w d\mu(w) \right) \wedge d^c \tau &\leq c \int_{|\zeta|=r} f^* \left(\left[\int_E K_{2+\epsilon}(x - w) d\mu(w) \right] \omega \right) \wedge d^c \tau \\ &\leq c \int_{|\zeta|=r} f^* (V_\mu(x) \omega) \wedge d^c \tau \leq \int_{|\zeta|=r} f^* \omega \wedge d^c \tau. \end{aligned}$$

Since

$$T_1(r) = \int_0^r \left\{ \int_{s=0}^t \left(\int_{|\zeta|=s} f^* \omega \wedge d^c \tau \right) ds \right\} d \log t$$

we finally obtain $\int_{|\zeta|=r} f^*(\int_E \Lambda_w d\mu(w)) \wedge d^c \tau \leq rT_1'(r)$, where $T_1'(r) = dT_1(r)/dr$. \square

PROOF OF THEOREM. The FMT states $N(w, r) + m(w, r) = T_2(r) + S(w, r) + C$.

(*) Hence $T_2(r) \leq N(w, r) + m(w, r) + C$. Suppose there exists a set of positive $2 + \varepsilon$ capacity $E \subset \mathbb{C}^2$ such that f takes on values in E only finitely many times. Let $E_m \subset E$ be the set such that f takes on values in E_m at most m times. Then $E = \cup E_m$. Since E has positive $2 + \varepsilon$ capacity one of the E_m , say E_{m_0} , must have positive capacity. Let μ be the equilibrium measure on E_{m_0} normalized such that $\mu(E_{m_0}) = 1$. Integrating (*) with respect to $d\mu(w)$ over \mathbb{C}^2 we have:

$$\begin{aligned} T_2(r) &\leq \int_{\mathbb{C}^2} N(w, r) d\mu(w) + \int_{\mathbb{C}^2} m(w, r) d\mu(w) + C \\ &= \int_{E_{m_0}} \left(\int_0^r n(w, t) d \log t \right) d\mu(w) + \int_{E_{m_0}} m(w, r) d\mu(w) + C, \\ T_2(r) &\leq m_0 \log r + cT''(r) + C. \end{aligned}$$

Since $\lim_{r \rightarrow \infty} [T''(r)/T_2(r)] = 0$ and $\lim_{r \rightarrow \infty} [\log r/T_2(r)] = 0$ we have a contradiction. Hence no such set E exists. \square

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