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A FURTHER REFINEMENT FOR COEFFICIENT ESTIMATES OF UNIVALENT FUNCTIONS

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ABSTRACT. The coefficient inequalities of FitzGerald are used to show that if $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is analytic and univalent in the unit disc, then $|a_n| < (1.0657)n$. The technique used to obtain this bound cannot yield a result better than $|a_n| < (1.0599)n$.

Let

$$S = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ is analytic and univalent in the unit disc} \right\}.$$

The Bieberbach Conjecture asserts that if $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is in S, then $|a_n| \leqslant n$ $(n = 2, 3, 4, \dots)$. This conjecture is the most renowned in geometric function theory, and much research and investigation has been devoted to establishing its validity. It has long been known **[10]** that the Bieberbach Conjecture is of the correct order in n, i.e. that there is some $C < \infty$ such that $|a_n| < Cn$ $(n = 2, 3, 4, \dots)$. Table 1 outlines the history of successive attempts to improve the number C.

Name	Year	$\|a_n\| < Cn$	
Littlewood **[10]**	1923	$\|a_n\| < en$	$\simeq 2.7183n$
Landau **[7]**	1929	$\|a_n\| < \left(\frac{1}{2} + \frac{1}{\pi}\right)en$	$\simeq 2.2244n$
Goluzin **[4]**	1946	$\|a_n\| < \frac{3}{4}en$	$\simeq 2.0388n$
Bazilevič **[1]**	1947	$\|a_n\| < \frac{9}{4}\left(\frac{1}{\pi}\int_0^{\pi}\frac{\sin x\,dx}{x} + .2649\right)n$	$\simeq 1.9240n$
Milin **[8]**, **[9]**	1949	$\|a_n\| < \frac{1}{2}en + 1.80$	$\simeq 1.3592n + 1.80$
Bazilevič **[2]**	1949	$\|a_n\| < \frac{1}{2}en + 1.51$	$\simeq 1.3592n + 1.51$
Milin **[11]**	1964	$\|a_n\| < \frac{(e^{1.6} - 1)^{1/2}}{1.6}n$	$\simeq 1.2427n$
FitzGerald **[3]**	1971	$\|a_n\| < \left(\frac{7}{6}\right)^{1/2}n$	$\simeq 1.0802n$
Horowitz **[6]**	1975	$\|a_n\| < \left(\frac{209}{140}\right)^{1/6}n$	$\simeq 1.0691n$

TABLE 1

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FitzGerald [3] has suggested the strongest method to date for improving the number C. Subsequently this author [6] applied FitzGerald's technique to improve the coefficient estimate. In this paper, FitzGerald's method is refined once again, and the resulting improvement is stated in the following

THEOREM. *If $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is in S, then*

$$|a_n| \leqslant \left(\frac{1,659,164,137}{681,080,400}\right)^{1/14} n < 1.0657n \qquad (n = 2, 3, 4, \dots). \tag{1}$$

The proof of the theorem is based upon the following reformulation of FitzGerald's Coefficient Inequality [3].

LEMMA. *If $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is in S, $\lambda_1, \lambda_2, \dots \lambda_L$ are complex numbers, and $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_L$ are positive integers, then*

$$\left|\sum_{k=1}^{L} \lambda_k |a_k|^2\right|^2 \leqslant \sum_{k=1}^{2L} |a_k|^2 \sum_{j=[(k+1)/2]}^{L} \sum_{l=|k-j|}^{j} \beta(j, l)(l - |j - k|)\mathrm{Re}(\lambda_j \overline{\lambda_l}) \tag{2}$$

where

$$\beta(j, l) = \begin{cases} 1 & \text{if } j = l, \\ 2 & \text{if } j \neq l \end{cases}$$

and $[x]$ denotes the greatest integer $\leqslant x$ by standard notation.

Inequality (2) differs from FitzGerald's original inequality in that (i) the order of summation on the right-hand side has been changed, and (ii) it has been generalized to include *complex* coefficients $\{\lambda_j\}$. The proof of this lemma involves a straightforward counting argument [5] and is omitted here.

PROOF OF THEOREM. In (2) set $L = n$, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$, and $\lambda_n = 1$ to obtain

$$|a_n|^4 \leqslant \sum_{k=1}^{2n} \Lambda_1(n, k)|a_k|^2 \tag{3}$$

where

$$\Lambda_1(n, k) = n - |n - k| \qquad (k = 1, 2, 3, \dots, 2n). \tag{4}$$

Next in (2) let $L = 2n$ and $\lambda_k = \Lambda_1(n, k)$ for $k = 1, 2, 3, \dots, 2n$. Then the left-hand side of (2) will have the form of the right-hand side of (3), and therefore it follows that

$$|a_n|^8 \leqslant \sum_{k=1}^{4n} \Lambda_2(n, k)|a_k|^2 \tag{5}$$

where

$$\Lambda_2(n, k) = \sum_{j=[(k+1)/2]}^{2n} \sum_{l=|k-j|}^{j} \beta(j, l)(l - |j - k|)\Lambda_1(n, j)\Lambda_1(n, l) \qquad (k = 1, 2, 3, \dots, 4n). \tag{6}$$

Now in (2) let $L = 4n$ and $\lambda_k = \Lambda_2(n, k)$ for $k = 1, 2, 3, \ldots, 4n$. Then the left-hand side of (2) will have the form of the right-hand side of (5), and therefore it follows that

$$|a_n|^{16} \leqslant \sum_{k=1}^{8n} \Lambda_3(n, k)|a_k|^2 \tag{7}$$

where

$$\Lambda_3(n, k) = \sum_{j=[(k+1)/2]}^{4n} \sum_{l=|k-j|}^{j} \beta(j, l)(l - |j - k|)\Lambda_2(n, j)\Lambda_2(n, l)$$

$$(k = 1, 2, 3, \ldots, 8n). \tag{8}$$

In fact it is easy to continue this process and establish for any positive integer m

$$|a_n|^{2^{m+1}} \leqslant \sum_{k=1}^{2^m n} \Lambda_m(n, k)|a_k|^2 \tag{9}$$

where $\Lambda_m(n, k)$ is defined recursively by

$$\Lambda_1(n, k) = n - |n - k| \qquad (k = 1, 2, 3, \ldots, 2n)$$

$$\Lambda_r(n, k) = \sum_{j=[(k+1)/2]}^{2^{m-1}n} \sum_{l=|k-j|}^{j} \beta(j, l)(l - |j - k|)\Lambda_{r-1}(n, j) \cdot \Lambda_{r-1}(n, l)$$

$$(k = 1, 2, 3, \ldots, 2^m n;\ r = 2, 3, \ldots,). \tag{10}$$

To establish (1) suppose that

$$C = \sup_n \sup_{f \in S} \left\{ \frac{|a_n|}{n} \right\}. \tag{11}$$

If $\varepsilon > 0$, then there is a positive integer n and a function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ in S such that

$$n(C - \varepsilon) < |a_n|. \tag{12}$$

From (7), (11), and (12) it follows that

$$n^{16}(C - \varepsilon)^{16} \leqslant C^2 \sum_{k=1}^{8n} \Lambda_3(n, k)k^2. \tag{13}$$

The sum in the right-hand side of (13) is too lengthy to be calculated by hand, but it only involves sums of the form $\sum_{k=1}^{s} k^t$ where s and t are positive integers. All of these sums are polynomials in s and t with rational coefficients. Thus closer examination discloses that the sum on the right-hand side of (13) is a polynomial $P(n)$ of degree 16 with rational coefficients. Moreover whenever n is a positive integer, then so is $P(n)$. A computer program was written to evaluate $P(n)$ for 17 different positive integers, and the resulting values were combined with the Lagrange Interpolation Formula **[12]** to derive that

$$
\begin{aligned}
P(n) = {} & \frac{1{,}659{,}164{,}137}{681{,}080{,}400} n^{16} \\
& - \frac{24{,}078{,}479}{14{,}968{,}800} n^{14} + \frac{5{,}621{,}807}{14{,}968{,}800} n^{12} \\
& - \frac{2{,}102{,}099}{9{,}525{,}600} n^{10} + \frac{94{,}789}{1{,}360{,}800} n^{8} \\
& - \frac{200{,}887}{7{,}484{,}400} n^{6} + \frac{87{,}797}{6{,}810{,}804} n^{4} \\
& - \frac{128}{10{,}395} n^{2}.
\end{aligned} \tag{14}
$$

By examining the terms in (14) pairwise it can be shown that

$$
P(n) \leqslant \frac{1{,}659{,}164{,}137}{681{,}080{,}400} n^{16} \quad \text{for } n \geqslant 2. \tag{15}
$$

Since $\varepsilon > 0$ was chosen arbitrarily, it follows from (13) and (15) that

$$
C^{14} \leqslant \frac{1{,}659{,}164{,}137}{681{,}080{,}400}
$$

whence

$$
C \leqslant \left(\frac{1{,}659{,}164{,}137}{681{,}080{,}400} \right)^{1/14} < 1.0657.
$$

This proves (1). □

It should be noted that the estimate (1) can continually be improved by appealing to (9) with ever-increasing values of m. However the number of calculations becomes prohibitive while the refinements become minuscule. For example if $m = 4$ and $n = 16$, then (9) becomes

$$
|a_{16}|^{32} \leqslant \sum_{k=1}^{256} \Lambda_4(16, k)|a_k|^2. \tag{16}
$$

Any estimate that can be obtained on the right-hand side of (16) cannot be better than that which would follow by replacing $|a_k|$ by k in this expression, since the Koebe function

$$
f(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k
$$

is in S. By doing so, the right-hand side of (16) sums to

$$
21918914609103152010877929180699798067 20 > (16)^{32}(1.0599)^{32}
$$

from which it follows that the improved estimate described above can be no better than

$$
|a_n| < 1.0599n.
$$

Thus while the Bieberbach Conjecture might still be a consequence of (2), any subsequent attempts to establish it by solely using the above iterative technique will be fruitless.

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