A FURTHER REFINEMENT FOR COEFFICIENT ESTIMATES OF UNIVALENT FUNCTIONS

DAVID HOROWITZ

ABSTRACT. The coefficient inequalities of FitzGerald are used to show that if 
\( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) is analytic and univalent in the unit disc, 
then \( |a_n| < (1.0657)n \). The technique used to obtain this bound cannot yield 
a result better than \( |a_n| < (1.0599)n \).

Let 
\[ S = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n : f \text{ is analytic and univalent in the unit disc} \right\}. \]

The Bieberbach Conjecture asserts that if \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) is in \( S \), then \( |a_n| < n \) (\( n = 2, 3, 4, \ldots \)). This conjecture is the most renowned in geometric function theory, and much research and investigation has been devoted to establishing its validity. It has long been known [10] that the Bieberbach Conjecture is of the correct order in \( n \), i.e. that there is some \( C < \infty \) such that \( |a_n| < Cn \) (\( n = 2, 3, 4, \ldots \)). Table 1 outlines the history of successive attempts to improve the number \( C \).

| Name          | Year | \( |a_n| < Cn \)        |
|---------------|------|------------------------|
| Littlewood [10]| 1923 | \( |a_n| < en \)           |
| Landau [7]    | 1929 | \( |a_n| < \left( \frac{1}{2} + \frac{1}{\pi} \right)en \) |
| Goluzin [4]   | 1946 | \( |a_n| < \frac{3}{4}en \) |
| Bazilevič [1] | 1947 | \( |a_n| < \frac{9}{4} \left( \frac{1}{\pi} \int_0^\pi \frac{\sin x}{x} \, dx + .2649 \right)n \) |
| Milin [8], [9]| 1949 | \( |a_n| < \frac{1}{2}en + 1.80 \) |
| Bazilevič [2] | 1949 | \( |a_n| < \frac{1}{2}en + 1.51 \) |
| Milin [11]    | 1964 | \( |a_n| < \frac{\left( e^{1.6} - 1 \right)^{1/2}}{n} \) |
| FitzGerald [3]| 1971 | \( |a_n| < \left( \frac{7}{6} \right)^{1/2}n \) |
| Horowitz [6]  | 1975 | \( |a_n| < \left( \frac{209}{140} \right)^{1/6}n \) |

Table 1

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DAVID HOROWITZ

FitzGerald [3] has suggested the strongest method to date for improving the number C. Subsequently this author [6] applied FitzGerald's technique to improve the coefficient estimate. In this paper, FitzGerald's method is refined once again, and the resulting improvement is stated in the following

**Theorem.** If \( f(z) = z + a_2z^2 + a_3z^3 \) \( \ldots \) is in \( S \), then

\[
|a_n| < \left( \frac{1.659,164,137}{681,080,400} \right)^{1/14} n < 1.06571^{n-1} \quad (n = 2, 3, 4, \ldots).
\]

The proof of the theorem is based upon the following reformulation of FitzGerald's Coefficient Inequality [3].

**Lemma.** If \( f(z) = z + a_2z^2 + a_3z^3 \) \( \ldots \) is in \( S \), \( \lambda_1, \lambda_2, \ldots, \lambda_L \) are complex numbers, and \( n_1 < n_2 < \cdots < n_L \) are positive integers, then

\[
\sum_{k=1}^{L} \lambda_k |a_k|^2 \leq 2 \sum_{k=1}^{L} |a_k|^2 \sum_{j=1}^{\min \{k+1,2\}} \sum_{l=1}^{j} \beta(j, l)(l - |j - k|) \text{Re}(\lambda_j \overline{\lambda_l})
\]

where

\[
\beta(j, l) = \begin{cases} 
1 & \text{if } j = l, \\
2 & \text{if } j \neq l
\end{cases}
\]

and \([x]\) denotes the greatest integer \( \leq x \) by standard notation.

Inequality (2) differs from FitzGerald's original inequality in that (i) the order of summation on the right-hand side has been changed, and (ii) it has been generalized to include complex coefficients \( \{\lambda_j\} \). The proof of this lemma involves a straightforward counting argument [5] and is omitted here.

**Proof of Theorem.** In (2) set \( L = n \), \( \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0 \), and \( \lambda_n = 1 \) to obtain

\[
|a_n|^4 < \sum_{k=1}^{2n} \Lambda_1(n, k) |a_k|^2
\]

where

\[
\Lambda_1(n, k) = n - |n - k| \quad (k = 1, 2, 3, \ldots, 2n).
\]

Next in (2) let \( L = 2n \) and \( \lambda_k = \Lambda_1(n, k) \) for \( k = 1, 2, 3, \ldots, 2n \). Then the left-hand side of (2) will have the form of the right-hand side of (3), and therefore it follows that

\[
|a_n|^8 < \sum_{k=1}^{4n} \Lambda_2(n, k) |a_k|^2
\]

where

\[
\Lambda_2(n, k) = \sum_{j=1}^{2n} \sum_{l=|j-k|}^{j} \beta(j, l)(l - |j - k|) \Lambda_1(n, j) \Lambda_1(n, l)
\]

\( (k = 1, 2, 3, \ldots, 4n) \).
Now in (2) let \( L = 4n \) and \( \lambda_k = \Lambda_3(n, k) \) for \( k = 1, 2, 3, \ldots, 4n \). Then the left-hand side of (2) will have the form of the right-hand side of (5), and therefore it follows that

\[
|a_n|^{16} \leq \sum_{k=1}^{8n} \Lambda_3(n, k)|a_k|^2
\]  

(7)

where

\[
\Lambda_3(n, k) = \sum_{j=(k+1)/2}^{4n} \sum_{l=(k-j)}^{j} \beta(j, l)(l - |j - k|)\Lambda_2(n, j)\Lambda_2(n, l)
\]

\( (k = 1, 2, 3, \ldots, 8n) \).  

(8)

In fact it is easy to continue this process and establish for any positive integer \( m \)

\[
|a_n|^{2m+1} \leq \sum_{k=1}^{2^m n} \Lambda_m(n, k)|a_k|^2
\]  

(9)

where \( \Lambda_m(n, k) \) is defined recursively by

\[
\Lambda_1(n, k) = n - |n - k| \quad (k = 1, 2, 3, \ldots, 2n)
\]

\[
\Lambda_r(n, k) = \sum_{j=(k+1)/2}^{2^{m-r} n} \sum_{l=(k-j)}^{j} \beta(j, l)(l - |j - k|)\Lambda_{r-1}(n, j)\cdot\Lambda_{r-1}(n, l)
\]

\( (k = 1, 2, 3, \ldots, 2^m n; r = 2, 3, \ldots, ) \).  

(10)

To establish (1) suppose that

\[
C = \sup_n \sup_{f \in S} \left\{ \frac{|a_n|}{n} \right\}.
\]  

(11)

If \( \varepsilon > 0 \), then there is a positive integer \( n \) and a function \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) in \( S \) such that

\[
n(C - \varepsilon) < |a_n|.
\]  

(12)

From (7), (11), and (12) it follows that

\[
n^{16}(C - \varepsilon)^{16} \leq C^2 \sum_{k=1}^{8n} \Lambda_3(n, k)k^2.
\]  

(13)

The sum in the right-hand side of (13) is too lengthy to be calculated by hand, but it only involves sums of the form \( \sum_{k=1}^{s} k^t \) where \( s \) and \( t \) are positive integers. All of these sums are polynomials in \( s \) and \( t \) with rational coefficients. Thus closer examination discloses that the sum on the right-hand side of (13) is a polynomial \( P(n) \) of degree 16 with rational coefficients. Moreover whenever \( n \) is a positive integer, then so is \( P(n) \). A computer program was written to evaluate \( P(n) \) for 17 different positive integers, and the resulting values were combined with the Lagrange Interpolation Formula [12] to derive that
\[ P(n) = \frac{1,659,164,137}{681,080,400} n^{16} - \frac{24,078,479}{14,968,800} n^{14} + \frac{5,621,807}{14,968,800} n^{12} - \frac{2,102,099}{9,525,600} n^{10} + \frac{94,789}{1,360,800} n^{8} - \frac{200,887}{7,484,400} n^{6} + \frac{87,797}{6,810,804} n^{4} - \frac{128}{10,395} n^{2}. \]  

(14)

By examining the terms in (14) pairwise it can be shown that

\[ P(n) < \frac{1,659,164,137}{681,080,400} n^{16} \quad \text{for } n > 2. \]  

(15)

Since \( \varepsilon > 0 \) was chosen arbitrarily, it follows from (13) and (15) that

\[ C^{14} < \frac{1,659,164,137}{681,080,400} \]

whence

\[ C < \left( \frac{1,659,164,137}{681,080,400} \right)^{1/14} < 1.0657. \]

This proves (1). □

It should be noted that the estimate (1) can continually be improved by appealing to (9) with ever-increasing values of \( m \). However the number of calculations becomes prohibitive while the refinements become minuscule. For example if \( m = 4 \) and \( n = 16 \), then (9) becomes

\[ 2^{56} W^{32} < 2^{56} A(16, 16) K_{2}^{2}. \]

(16)

Any estimate that can be obtained on the right-hand side of (16) cannot be better than that which would follow by replacing \( |a_{k}| \) by \( k \) in this expression, since the Koebe function

\[ f(z) = \frac{z}{(1 - z)^{2}} = \sum_{k=1}^{\infty} k z^{k} \]

is in \( S \). By doing so, the right-hand side of (16) sums to

\[ 2191891460910315201087792918069979806720 > (16)^{32}(1.0599)^{32} \]

from which it follows that the improved estimate described above can be no better than

\[ |a_{n}| < 1.0599n. \]

Thus while the Bieberbach Conjecture might still be a consequence of (2), any subsequent attempts to establish it by solely using the above iterative technique will be fruitless.

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