AN EXTREMAL PROBLEM FOR QUASICONFORMAL MAPPINGS IN AN ANNULUS

ALVIN M. WHITE

Abstract. The following extremal problem is solved. We consider a family of continuosly differentiable univalent quasiconformal mappings \( w = f(z) \) of the annulus \( r < |z| < 1 \) onto the unit disk minus some continuum containing the origin. For a point \( b \) on a fixed circle, maximize \( |f(b)| \) within the family.

The problem is solved by using a variational method due to Schiffer. The extremal function and the maximum are found in terms of the Weierstrass \( \wp \)-function and the elliptic modular function.

In his investigation of numerical construction of conformal mappings, D. Gaier \[3\] considered the family \( F \) of functions \( f(z) \) regular, analytic and univalent in the annulus \( R: r < |z| < 1 \) and satisfying the following three conditions:

\[
|f(z)| < 1 \quad \text{in } R \quad \text{while} \quad |f(z)| = 1 \quad \text{on } |z| = 1; \tag{1}
\]

that is, \( f(z) \) maps \( R \) onto the unit disk minus some continuum \( \Gamma \);

\[
f(z) \neq 0 \quad \text{in } R, \quad (\Gamma \text{ contains the origin}); \tag{2}
\]

\[
f(1) = 1, \quad (\text{a normalization}). \tag{3}
\]

Gaier raised the question of finding the maximum of \( |f(z) - z| \) for all \( f \in F \) and \( z \in R \). Duren and Schiffer \[2\] solve this problem by a method of variation within the family \( F \). (See also Gehring and Hällström \[5\] and Gaier and Huckemann \[4\].)

In order to answer the question raised by Gaier, Duren and Schiffer first solve the extremal problem:

\[
\max |f(b)|, \quad f \in F, \quad \text{where } |b| = r_1, \quad r < r_1 < 1. \tag{*}
\]

This problem is of independent interest. Grötzsch \[6\] previously solved it by the method of extremal length.

In the present paper the requirement of analyticity is weakened and problem (\*\*) is solved within a family \( F' \) of quasiconformal mappings by a variational method.

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1. Let \( w = f(z) = u(x, y) + iv(x, y) \) be a family of continuously differentiable univalent mappings of a domain \( D_z \) in the \( z \) plane onto a domain \( D_w \) in the \( w \) plane.

We call the mapping \( K\)-quasiconformal in \( D_z \) if it satisfies the inequality\(^2\)

\[
|f_z|^2 + |f_{\bar{z}}|^2 < K(|f_z|^2 - |f_{\bar{z}}|^2); \quad K > 1,
\]

\[
\left| \frac{f_z}{f_{\bar{z}}} \right|^2 < \frac{K - 1}{K + 1}
\]

for all points of \( D_z \). Note that

\[
|f_z|^2 + |f_{\bar{z}}|^2 = \frac{1}{2} (u_x^2 + u_y^2 + v_x^2 + v_y^2)
\]

and

\[
|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x = \frac{\partial (u, v)}{\partial (x, y)}
\]

is the Jacobian of the mapping \([1]\).

Denote the family \( f(z) \) of univalent, class \( C^1 \) functions satisfying (1), (2), (3), (4) as \( F' \). Consider the extremal problem:

\[
\max |f(b)|, \quad f \in F' \quad \text{where} \quad |b| = r_1, \quad r < r_1 < 1.
\]

Problem (7) is solved by a variational method. In the variational method we seek that function (if it exists) in the family for which the functional \( \chi[f] = |f(b)| \) attains its maximum value. The value of the functional is compared for functions of the family which are near each other in a certain sense. A comparison function within the family \( F' \) is constructed by a method of Schiffer \([12], [2]\). The existence of the extremal function can be shown as in \([12], [10]\).

Let \( w = f(z) \) be a given function in \( F' \) and let \( \Gamma \) denote the complement within the open disk of the range \( \Delta(f) \) of \( f \). Let \( w_0 \neq 0 \) be a fixed point on the continuum \( \Gamma \). Corresponding to each point \( w_0 \) it is known \([2]\) that there exist variation functions \( V_\rho(w) \) analytic and univalent in the range of \( f \) having the following properties:

(i) \( \lim_{\rho \to 0} V_\rho(w) = w; \)
(ii) \( |V_\rho(w)| = 1 \) for \( |w| = 1; \)
(iii) \( V_\rho(w) \neq 0 \) for \( w \) in \( \Delta(f) \);
(iv) \( V_\rho(1) = 1. \)

Such a function is

\[
V(w) = w \left[ 1 + \frac{\alpha \rho^2 (1 - w)}{(w - w_0)(1 - w_0)w_0} + \frac{\bar{\alpha} \rho^2 (1 - w)}{(1 - \bar{w}_0w)(1 - \bar{w}_0)\bar{w}_0} + 0(\rho^3) \right]
\]

for proper choice of the coefficient \( \alpha \). A comparison function in the class \( F' \) is

\[
f^\rho(z) = V_\rho[f(z)].
\]

\(^2\)This is not the currently customary definition.
Suppose \( f(z) \) is the extremal function, and denote \( f(b) = B \). We have \( |V_\rho[f(b)]| < |f(b)| \). Introducing (8) this yields

\[
1 + \frac{a \rho^2(1 - B)}{(B - w_0)(1 - w_0)w_0} + \frac{\bar{a} \rho^2(1 - B)}{(1 - \bar{w}_0 B)(1 - \bar{w}_0)\bar{w}_0} + O(\rho^3) < 1. \tag{10}
\]

Noting that \( \Re\{a\} = \Re\{\bar{a}\} \), and \( |B| < 1 \), a calculation yields

\[
\Re\left\{ \frac{a \rho^2}{w_0(B - w_0)(1 - w_0)\bar{B}} + O(\rho^3) \right\} < 0. \tag{11}
\]

2. From the fundamental lemma of boundary variation [11] in the theory of conformal mapping, we can conclude from inequality (11), which is valid for all admissible variations, that \( \Gamma \) is an analytic arc \( w = w(t) \) satisfying the differential equation

\[
\frac{(w')^2}{w(B - w)(1 - w\bar{B})} > 0. \tag{12}
\]

The analytic arc \( \Gamma \) must pass through the origin since \( f(z) \neq 0 \) in \( R \).

Obviously the ray \( w = Bt, 0 \leq t \leq 1 \), satisfies (12) and contains the origin. Because of the uniqueness theorem for the differential equation [2], this segment is the only solution containing the origin and hence \( \Gamma \) must be a part of it.

Thus we prove that the extremal function \( f \) maps the annulus \( R \) onto the unit disk slit along a segment starting at the origin in the direction of \( B \).

3. Without loss of generality we can assume that \( B > 0 \). Indeed, let \( B = |B| e^{i\beta} \) and let \( \alpha \) be defined by \( f(e^{i\alpha}) = e^{i\beta} \). Then the function \( \varphi(z) = e^{-i\beta} f(e^{i\alpha}z) \) also belongs to \( F' \) and maps \( R \) onto the disk slit along the real axis and takes the value \( |B| \) on the circle \( |z| = |b| \).

4. In order to characterize the extremal function \( f(z) \) another variation will be introduced. We choose a point \( w_0 \) in the range of \( f \) and define a variation \( \tilde{V}_\rho(w) \) which has the form (8) for \( |w - w_0| > \rho \), but must be modified inside the disk to remain of class \( C^1 \).

\[
\tilde{V}_\rho(w) = w \left[ 1 + \frac{a(1 - w)}{w_0(1 - w_0)} \cdot \left( w - w_0 \right) \left( 2 - \frac{|w - w_0|^2}{\rho^2} \right) \right.
\]

\[
+ \frac{\bar{a} \rho^2(1 - w)}{(1 - \bar{w}_0 w)(1 - \bar{w}_0)\bar{w}_0} + O(\rho^3) \right] \tag{13}
\]

satisfies the requirement. An easy calculation shows that our functional \( \chi[f] = |f(b)| \) satisfies the asymptotic equation
\[ \chi[f^*] = \chi[f] \left[ 1 + \text{Re} \left( \frac{a \rho^2(1 - B^2)}{w_0(B - w_0)(1 - B w_0)} + O(\rho^3) \right) \right] \]  

since we may assume that for sufficiently small \( \rho \), \( B \) is outside the disk \( |w - w_0| < \rho \).

The assumption that \( f(z) \) is the extremal function for our problem implies that \( \chi[f^*] < \chi[f] \) for all variations \( \tilde{V}_\rho[w] \) which preserve the \( K \)-quasiconformality of the varied function. This condition allows the application of [12]. It is seen that the extremal function \( w = f(z) \) satisfies the relation

\[ \frac{J(w)}{J(w)} = \sqrt{1 - \frac{K}{K + 1}} J(w) = k(z) \]  

where

\[ J(w) = \int_1^w \frac{dw}{\sqrt{w(B - w)(1 - wB)}} \]  

and \( k(z) \) is analytic. Note that \( k(z) \) is multivalued and has a singularity at \( b \).

The extension (13) of (8) into \( |w - w_0| < \rho \) may lack a uniformity property that is needed in [12] as was pointed out by Renelt [10]. This is remedied by using the solution of a Beltrami equation as shown in [14]. The rest of the procedure remains unchanged.

We can transform (15) so as to express \( J(w) \) in terms of \( k(z) \) and \( k(\bar{z}) \).

\[ J(w) = \frac{K + 1}{2} k(z) + \frac{1}{2} \sqrt{K^2 - 1} \ \frac{1}{k(z)}. \]  

Differentiating (15') with respect to \( x \) yields

\[ \frac{1}{\sqrt{w(B - w)(1 - wB)}} f_x = \frac{K + 1}{2} k'(z) + \frac{1}{2} \sqrt{K^2 - 1} k'(\bar{z}). \]  

Let us consider that point on \( |z| = r \) which corresponds to the right endpoint of the slit \( \Gamma \). Obviously, at that point the Jacobian of the mapping vanishes and hence \( f_x = 0 \). This means that \( k'(z) \) has a zero on the circle \( |z| = r \).

5. In order to study the boundary behavior of the analytic function \( k(z) \), we observe that the circumference \( |z| = 1 \) is mapped by \( f(z) \) onto \( |w| = 1 \). On \( |w| = 1 \) we can write

\[ dJ(w) = \frac{dw}{\sqrt{w(B - w)(1 - wB)}} = \frac{ie^{ia}da}{\sqrt{e^{ia}(B - e^{ia})(1 - Be^{ia})}} \]

where \( w = e^{ia} \). Hence

\[ dJ(w) = \frac{id\alpha}{\sqrt{(B - e^{ia})(e^{-ia} - B)}} = \frac{da}{|B - e^{ia}|} = \text{real}. \]

In view of (15) we can assert that \( dk(z) \) is real for \( |z| = 1 \).

Next consider the image of \( |z| = r \) which we have shown to be a segment of the positive real axis. Since this segment does not reach the point \( B \) which
corresponds to $b$ ($r < |b| < 1$), $J(w)$ is real on that segment and so is $k(z)$ in view of (15).

Since we can parametrize $z = e^{ia}$ and $z = re^{ia}$ on the two circumferences, we have in both cases $dz = iz\, da$ and from the observation that $k'(z)dz = \text{real}$ in each case, we infer that on each circumference

$$l(z) = z^2k'(z)^2 \leq 0. \quad (17)$$

From the definition of $k(z)$ it is easily seen that $k'(z)$ is unbounded in the neighborhood of $z = b$, while $\sqrt{z} - b\, k'(z)$ is regular.

Since $l(z)$ is real for $|z| = r$ and $|z| = 1$, by the Schwarz reflection principle, we have

$$l(1/\bar{z}) = \overline{l(z)} = l(r^2/\bar{z}).$$

Hence $l(r^2z) = l(z)$. Now, let $t = \log z$, $L(t) = l(z)$. Then, in the rectangle

$$\log r < \text{Re}\{t\} < \log 1/r, \quad 0 < \text{Im}\{t\} < 2\pi$$

the function $L(t)$ is regular analytic except for two simple poles at $t_1 = \log b$ and $t_2 = \log 1/b$. It is doubly periodic

$$L(t + 2\log r) = L(t), \quad L(t + 2\pi i) = L(t).$$

In view of (17), $L(t)$ is negative for $\text{Re}\{t\} = 0$ and $\text{Re}\, t = \log r$. Consider the half period parallelogram

$$\log r < \text{Re}\{t\} < 0, \quad \gamma < \text{Im}\, t < \gamma + \pi$$

where $b = |b|e^{i\gamma}$. We can map this parallelogram onto the upper half-plane such that the point $\log b$ on its boundary goes into infinity. The mapping function $\psi(t)$ is easily seen to be doubly periodic with the same periods as $L(t)$ and with the same poles. Hence, by the well-known uniqueness theorem for elliptic functions

$$L(t) = A\psi(t) + C.$$

Since $\psi(t)$ and $L(t)$ are both real for $\text{Re}\{t\} = 0$ and $\text{Re}\{t\} = \log r$, clearly $A$ and $C$ are real constants. We also see that $L(t)$ gives a one-to-one map of the boundary of the half period parallelogram onto the real axis. Hence, it has precisely one simple zero which corresponds to the zero of $k'(z)$ on the circle $|z| = r$. We know by (17) that $l(z)$ is nonpositive for those $z$ corresponding to the vertical lines of the parallelogram. Hence, its simple zero must be a corner point: either $[\log r + i\gamma]$ or $[\log r + i(\gamma + \pi)]$. Observe that the only other point on the boundary where $L(t)$ can change its sign is the point $\log b$ where it has a simple pole. Thus we infer that on the boundary of the half period parallelogram $L(t)$ is positive on the segment $[\log r + i\gamma, \log |b| + i\gamma]$ and is negative on the rest.

Returning to the $z$-plane, we find that $z^2k'(z)^2$ is negative for $z = se^{i\gamma}$ with $|b| < s < 1$. Now,

$$\frac{d}{ds} k(e^{i\gamma}s) = e^{i\gamma}k'(e^{i\gamma}s)$$

and hence we can conclude that $d/\, ds\, k(e^{i\gamma}s)$ is imaginary for this interval.
Thus $k(b) - k(e^{\gamma}) = \text{imaginary}$. On the other hand, observe that $J(B)$ is pure imaginary and so is, by (15), $k(b)$. Hence, we conclude $k(e^{\gamma}) = \text{imaginary}$. Finally, $J(w)$ is real for $|w| = 1$ by (16), whence $k(e^{\gamma}) = 0$. By (15') the image $w$ of $e^{\gamma}$ satisfies $J(w) = 0$. By (16), $J(1) = 0$. By the normalization, the point 1 corresponds to 1. Hence, $e^{\gamma} = 1$, and we infer $b > 0$.

It has been shown that $L(t)$ vanishes at $\log r + i\gamma$ and that $\gamma = 0$. This implies that $k'(r) = 0$.

6. Now we complete the argument as follows: $k(z)$ is pure imaginary on the real axis between $b$ and 1. It follows from (15') and (16) that $w = f(z)$ is real on that segment. On the segment between $r$ and $b$ we see that $l(z) > 0$, which implies $(k'(z)z)^2 > 0$. Hence $k'(z)z$ and $k'(z)$ are real on that segment. Thus (15'') leads to the differential equation

$$\frac{dw}{\sqrt{w(B-w)(1-wB)}} = \text{real}.$$ 

This implies that $w = f(z)$ is real between $r$ and $b$.

We conclude that $f(z)$ maps the inner circumference $|z| = r$ onto the continuum $\Gamma$ which is a segment of the positive axis. The point $z = -r$ is mapped into $w = 0$ and $z = r$ into the right end of the slit.

Thus the single valued analytic function $[zk'(z)]^2$ is negative on the circumferences $|z| = 1$ and $|z| = r$, has a simple pole at $z = b$, and vanishes at $z = r$.

By Rouché's theorem it is easy to see that this function takes every nonnegative value precisely once. Hence it is a univalent function in the circular ring which maps this domain onto the complex plane slit along two segments of the negative axis. In particular, the segment corresponding to the circumference $|z| = r$ begins at the origin.

The geometric properties just described identify $[zk'(z)]^2$ up to a factor as

$$[zk'(z)]^2 = \frac{A}{\varphi(\theta) - \varphi(\phi)},$$

where $\theta = \log z/r$ and $\phi = \log b/r$ and $\varphi(\theta)$ is the Weierstrass $\wp$-function of period $\log 1/r$ and $i\pi$ (see, for example, [7, p. 191]).

7. Solving for $J$ yields

$$J[f(z)] = \frac{K+1}{2} [k + \bar{k}Q]$$

where $J$ is the elliptic integral defined in (16) and $Q = \sqrt{(K-1)/(K+1)}$.

After inverting (19) we will have an explicit expression for the extremal function

$$f(z) = P \left\{ \frac{K+1}{2} (k(z) + \overline{k(z)} Q) \right\},$$

where $P$ is closely related to the Weierstrass $\wp$-function.
8. Although the analytic function \( k(z) \) is determined only up to a factor, various relationships can be used in order to determine the extremal value \( B \). The extremal function is explicitly expressed in (20) as the inverse of the elliptic integral (16). The periods of (20) are functions of \( B \). The branch points of (16) are real.

From (1) it follows that as \( z \) traverses the unit circle, \( w \) does the same. By Cauchy’s integral theorem

\[
\int_{|w|=1} dw = 2 \int_0^B \frac{dw}{\sqrt{w(B - w)(1 - Bw)}} = 2\omega_1. \tag{21}
\]

The right-hand side of (21) is twice the real period of the elliptic integral (16). Consider equation (15) for \(|z| = 1\). Since \( dk \) is real on that circumference, we find that

\[
\int_{|z|=1} k'(z)dz = 2(1 - Q)\omega_1. \tag{22}
\]

The imaginary period is

\[
\omega_2 = \int_B^{1/B} \frac{dw}{\sqrt{w(B - w)(1 - Bw)}}. \tag{23}
\]

Since

\[
\int_B^1 \frac{dw}{\sqrt{w(B - w)(1 - Bw)}} = \int_1^{1/B} \frac{dw}{\sqrt{w(B - w)(1 - Bw)}},
\]

we have \( \frac{1}{2} \omega_2 = \int_B^1 \frac{dw}{\sqrt{w(B - w)(1 - Bw)}} \). From (15) we have

\[
\int_b^1 dk = \int_B^1 dJ - Q \int_B^1 \overline{dJ} = \frac{1}{2} \omega_2 + \frac{Q}{2} \omega_2, \tag{24}
\]

from which follows

\[
\int_b^1 k'(z)dz = \frac{1}{2} (1 + Q)\omega_2. \tag{25}
\]

The left sides of (22) and (25) are known up to a factor. The quotient of (25) by (22) is thus a completely determined function of \( b \). Denote the quotient by \( E(b) \). The ratio of the periods is then

\[
4E(b) \frac{(1 - Q)}{(1 + Q)} = \frac{\omega_2}{\omega_1} = \tau. \tag{26}
\]

9. Since the ratio \( \tau \) of the periods is known from (26), it is possible to express the extremal value \( B \) as a function of the fixed point \( b \). This can be done as follows.

Putting the elliptic integral (16) in the Weierstrass normal form, we have
\[ \int \frac{dw}{\sqrt{w(B - w)(1 - wB)}} = \int \frac{dw}{\sqrt{4(w - e_1)(w - e_2)(w - e_3)}} \]

where \( e_1 + e_2 + e_3 = 0 \).

The result of a straightforward calculation is

\[ B^2 = \frac{e_2 - e_3}{e_1 - e_3}. \]

But the right side is the elliptic modular function \( \lambda(\tau) \) which is a single valued function of \( \tau \), [15], [16], which in turn is a function of \( b \).

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DEPARTMENT OF MATHEMATICS, HARVEY MUD COLLEGE, CLAREMONT, CALIFORNIA 91711