A NECESSARY AND SUFFICIENT CONDITION
FOR BLOCH FUNCTIONS

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Abstract. A necessary and sufficient condition is given for a subset $E \subseteq \mathbb{C}$ to satisfy

$$\text{Sup}\{ |f'(z)| (1 - |z|^2)z \in f^{-1}(E) \} < \infty$$

$$\Rightarrow \text{Sup}\{ |f'(z)| (1 - |z|^2)z \in D \} < \infty$$

when $f : D \to \mathbb{C}$ is analytic. The condition is that the complement of $E$ should not contain large discs.

A Bloch function on $D = \{ z \in \mathbb{C} | |z| < 1 \}$ is an analytic function $f : D \to \mathbb{C}$ satisfying $\text{Sup}\{ |f'(z)| (1 - |z|^2)z \in D \} < \infty$. See [1] for other characterizations and some interesting properties of Bloch functions.

The following theorem is the main result of this note.

**Theorem 1.** Suppose $E \subseteq \mathbb{C}$. Then the condition

$$\text{Sup}\{ |f'(z)| (1 - |z|^2)z \in f^{-1}(E) \} < \infty$$

is a sufficient condition for an analytic function $f : D \to \mathbb{C}$ to be a Bloch function if and only if the radii of the discs contained in $\mathbb{C} - E$ are bounded above.

For example $E = \mathbb{Z}^2 = \text{the set of Gaussian integers}$ works, but no finite set $E$ will work.

This theorem was motivated by a theorem of P. Lappan [3] about normal functions.

A meromorphic function $f$ on $D$ is said to be normal if

$$\text{Sup}\{ |f'(z)| (1 - |z|^2) / (1 + |f(z)|^2)z \in D \} < \infty$$

(see [1]).

**Theorem [Lappan 5-Value Theorem].** If $f$ is a meromorphic function defined on the unit disc $D$ and

$$\text{Sup}\{ |f'(z)|(1 - |z|^2) / (1 + |f(z)|^2)z \in f^{-1}(E) \} = M < \infty$$

for any subset $E$ of the Riemann sphere containing at least 5 points (3 finite points if $f$ is analytic) then $f$ is a normal function.

Since Bloch functions are closely related to normal functions it is natural to

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ask whether Lappan's theorem has an analogue for Bloch functions. Theorem 1 answers this question. The author thanks L. A. Rubel for calling his attention to the problem.

**Proof (Theorem 1).** Suppose that the radii of the discs contained in \( C - E \) are bounded by \( R \) and \( f: D \to C \) is an analytic function with 
\[
\text{Sup}\{|f'(z)|(1 - |z|^2)|z \in f^{-1}(E)\} = M < \infty.
\]

If \( f \) is not a Bloch function, then for each \( r > 0 \) there exists a schlicht disc \( \Delta = \{w \in C| |w - w_0| < r + R + 1\} \) in the range of \( f \) (i.e. \( f \) has a single-valued analytic inverse on \( \Delta \)—see [1]).

By hypothesis, \( \{w \in C| |w - w_0| < R + 1\} \subseteq C - E \) is false and so there exists \( w'_0 \in E \) with \( |w'_0 - w_0| < R + 1 \). Hence \( \{w \in C| |w - w'_0| < r\} \) is a schlicht disc in the range of \( f \). Thus there exists a 1-1 conformal mapping \( \phi: D \to D \) so that \( (f \circ \phi)(z) = w'_0 + rz \). Hence \( |f'(\phi(0))| |\phi'(0)| = r \). By the Schwarz Lemma (12.5.3 of [5]) 
\[
|f'(\phi(0))|(1 - |\phi(0)|^2) > r.
\]

Observe that \( \phi(0) \in f^{-1}(E) \), which means that a choice of \( r \) with \( r > M \) will contradict the supposition at the beginning of the proof. Thus \( f \) must be a Bloch function and the "if" part of the theorem is proved.

Notice that the above argument shows that every schlicht disc in the range of the Bloch function \( f \) has radius no larger than \( M + R \), and thus
\[
\text{Sup}\{|f'(z)|(1 - |z|^2)|z \in D\} < (M + R)/B
\]
where \( B \) is Bloch's constant.

To show the converse it must be shown that, if \( C - E \) contains discs of arbitrarily large radii, then there exists an analytic function \( f \) on \( D \) which is not a Bloch function but satisfies the condition (*)

It is easy to see that \( C - E \) must contain an infinite sequence of disjoint discs of the form \( D_n = \{w \in C| |w - w_n| < n\} \) with \( |w_{n+1}| > |w_n| + 2n + 1 \) for each \( n > 1 \). For each \( n \) it is possible to construct a narrow open channel \( G_n \) joining \( D_n \) to \( D_{n+1} \) so that the following conditions both hold.

(i) \( G = \bigcup_{n=1}^{\infty}(D_n \cup G_n) \) is simply connected.

(ii) \( G \) does not contain any disc of radius larger than \( 1 \) centered at any point of \( G_n \).

By the Riemann mapping theorem there exists a 1-1 onto conformal map \( f: D \to G \). Since \( f \) has the schlicht discs \( D_n \) in its range it cannot be a Bloch function.

Suppose \( z \in D \) and \( f(z) \in E \). Then \( f(z) \in G_n \) for some \( n \) since \( D_n \subseteq C - E \) for each \( n \). By applying the \( \frac{1}{4} \) theorem [5, Theorem 14.14] to the function \( s \mapsto f((s + z)/(1 + zs)) \) it follows that the range of \( f \) contains a (schlicht) disc of radius \( \frac{1}{4} |f'(z)|(1 - |z|^2) \) centered at \( f(z) \). Thus \( \frac{1}{4} |f'(z)|(1 - |z|^2) < 1 \) by (ii) and so 
\[
\text{Sup}\{|f'(z)|(1 - |z|^2)|z \in f^{-1}(E)\} < 4.
\]
The proof of the theorem is now complete.

**Remarks.** (1) If \( E \subseteq C \) is such that the radii of the discs contained in
C – E are bounded and f is a meromorphic function satisfying (\ast) then f must be analytic and thus a Bloch function (by Theorem 1).

(2) \( \mathcal{B}_0 \) is defined to be the set of Bloch functions satisfying the condition that \( \left| f'(z) \right| (1 - |z|^2) \to 0 \) as \( |z| \to 1 \). Since there exist bounded functions which are not in \( \mathcal{B}_0 \) (e.g. \( f(z) = w_0 + \delta \exp[(z + 1)/(z - 1)] \)) it follows that \( E \subseteq \mathbb{C} \) satisfies

\[
\left| f'(z) \right| (1 - |z|^2) \to 0 \quad \text{as} \quad |z| \to 1 \quad \text{with} \quad z \in f^{-1}(E) \Rightarrow f \in \mathcal{B}_0 \quad (\ast\ast)
\]

if and only if \( E \) is dense in \( \mathbb{C} \). (Any function omitting the values \( E \) satisfies (\ast\ast).)

(3) In conjunction with Lappan's 5-value theorem, it can be shown that for each subset \( E \) of the Riemann sphere where \( E \) contains at least 5 points there exists an increasing function \( \alpha_E : [0, \infty) \to (0, \infty) \) such that

\[
\sup \left\{ \left| f'(z) \right| (1 - |z|^2) / (1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} = M < \infty
\]

implies

\[
\sup \left\{ \left| f'(z) \right| (1 - |z|^2) / (1 + |f(z)|^2) \mid z \in D \right\} \leq \alpha_E (M).
\]

The proof given by Lappan of his 5-value theorem does not give the existence of \( \alpha_E \). However, by applying the lemma of [6] to the Moebius-invariant family

\[\mathcal{F} = \{ f \mid f \text{ meromorphic on } D \text{ and} \]

\[
\sup \left\{ \left| f'(z) \right| (1 - |z|^2) / (1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} \leq M \}
\]

in the same way that Lappan applied Theorem 1 of [4] it follows that \( \mathcal{F} \) is a normal family. Hence, by the Marty criterion [2, p. 158]

\[
\sup \left\{ \left| f'(0) \right| / (1 + |f(0)|^2) \mid f \in \mathcal{F} \right\} \leq \alpha_E (M) < \infty.
\]

Next, \( f \in \mathcal{F} \) implies \( f((z + a)/(1 + \bar{a}z)) \in \mathcal{F} \) for each \( a \in D \) and this implies \( \left| f'(a) \right| (1 - |a|^2) / (1 + |f(a)|^2) \leq \alpha_E (M) \) for each \( a \in D \) and for each \( f \in \mathcal{F} \).

This minor modification of Lappan's result may be used to obtain a version of the Lappan 5-value theorem for normal meromorphic functions defined on \( \mathbb{C} \).

A meromorphic function \( f \) on \( \mathbb{C} \) is said to be normal if \( \sup \left\{ \left| f'(z) \right| / (1 + |f(z)|^2) \mid z \in \mathbb{C} \right\} < \infty \) or equivalently if the family \( \{ f(e^{ia}z + a) \mid a \in \mathbb{C}, \alpha \in [0, 2\pi) \} \) is a normal family. The maps \( z \mapsto e^{ia}z + a \) are the conformal isometries of \( \mathbb{C} \) in the Euclidean metric. Another equivalent condition is that \( \{ f(z + a) \mid a \in \mathbb{C} \} \) be a normal family. These equivalences follow from the Marty criterion [2, p. 158].

**Theorem 2.** Let \( E \) be a subset of the Riemann sphere containing at least 5 points. Let \( f \) be a meromorphic function on \( \mathbb{C} \) satisfying

\[
\sup \left\{ \left| f'(z) \right| / (1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} = M < \infty.
\]
Then $f$ is a normal function and in fact $\sup\{|f'(z)|/(1 - |f(z)|^2)|z \in \mathbb{C}\} \leq \alpha_E(M)$.

**Proof.** Fix $z_0 \in \mathbb{C}$ and set $g(z) = f(z_0 + z)$ for $z \in D$. Then

$$|g'(z)|(1 - |z|^2)/(1 + |g(z)|^2) < |f'(z_0 + z)|/(1 + |f(z_0 + z)|^2) < M$$

for $z \in g^{-1}(E)$. Thus $|g'(0)|/(1 + |g(0)|^2) = |f'(z_0)|/(1 + |f(z_0)|^2) < \alpha_E(M)$ as required.

**References**


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