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A NECESSARY AND SUFFICIENT CONDITION FOR BLOCH FUNCTIONS

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ABSTRACT. A necessary and sufficient condition is given for a subset $E \subseteq \mathbf{C}$ to satisfy

$$\operatorname{Sup}\{|f'(z)|\,(1-|z|^2)|z \in f^{-1}(E)\} < \infty$$
$$\Rightarrow \operatorname{Sup}\{|f'(z)|\,(1-|z|^2)|z \in D\} < \infty$$

when f: $D \to \mathbf{C}$ is analytic. The condition is that the complement of E should not contain large discs.

A Bloch function on $D = \{z \in \mathbf{C}|\ |z| < 1\}$ is an analytic function f: $D \to \mathbf{C}$ satisfying $\operatorname{Sup}\{|f'(z)|\,(1-|z|^2)|z \in D\} < \infty$. See [1] for other characterizations and some interesting properties of Bloch functions.

The following theorem is the main result of this note.

THEOREM 1. *Suppose $E \subseteq \mathbf{C}$. Then the condition*

$$\operatorname{Sup}\{|f'(z)|\,(1-|z|^2)|z \in f^{-1}(E)\} < \infty \tag{$*$}$$

is a sufficient condition for an analytic function f: $D \to \mathbf{C}$ to be a Bloch function if and only if the radii of the discs contained in $\mathbf{C} - E$ are bounded above.

For example $E = \mathbf{Z}^2$ = the set of Gaussian integers works, but no finite set E will work.

This theorem was motivated by a theorem of P. Lappan [3] about normal functions.

A meromorphic function f on D is said to be normal if

$$\operatorname{Sup}\{|f'(z)|\,(1-|z|^2)/(1+|f(z)|^2)|z \in D\} < \infty$$

(see [1]).

THEOREM [LAPPAN 5-VALUE THEOREM]. *If f is a meromorphic function defined on the unit disc D and*

$$\operatorname{Sup}\{|f'(z)|(1-|z|^2)/(1+|f(z)|^2)|z \in f^{-1}(E)\} = M < \infty$$

for any subset E of the Riemann sphere containing at least 5 points (3 finite points if f is analytic) then f is a normal function.

Since Bloch functions are closely related to normal functions it is natural to

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ask whether Lappan's theorem has an analogue for Bloch functions. Theorem 1 answers this question. The author thanks L. A. Rubel for calling his attention to the problem.

PROOF (THEOREM 1). Suppose that the radii of the discs contained in $\mathbf{C} - E$ are bounded by R and $f\colon D \to \mathbf{C}$ is an analytic function with

$$\operatorname{Sup}\{|f'(z)|(1 - |z|^2) | z \in f^{-1}(E)\} = M < \infty.$$

If f is not a Bloch function, then for each $r > 0$ there exists a schlicht disc $\Delta = \{w \in \mathbf{C} |\, |w - w_0| < r + R + 1\}$ in the range of f (i.e. f has a single-valued analytic inverse on Δ–see [1]).

By hypothesis, $\{w \in \mathbf{C} |\, |w - w_0| < R + 1\} \subseteq \mathbf{C} - E$ is false and so there exists $w_0' \in E$ with $|w_0' - w_0| < R + 1$. Hence $\{w \in \mathbf{C} |\, |w - w_0'| < r\}$ is a schlicht disc in the range of f. Thus there exists a 1-1 conformal mapping ϕ: $D \to D$ so that $(f \circ \phi)(z) = w_0' + rz$. Hence $|f'(\phi(0))|\,|\phi'(0)| = r$. By the Schwarz Lemma (12.5.3 of [5]) $|\phi'(0)| < 1 - |\phi(0)|^2$ and so

$$|f'(\phi(0))|(1 - |\phi(0)|^2) \geqslant r.$$

Observe that $\phi(0) \in f^{-1}(E)$, which means that a choice of r with $r > M$ will contradict the supposition at the beginning of the proof. Thus f must be a Bloch function and the "if" part of the theorem is proved.

Notice that the above argument shows that every schlicht disc in the range of the Bloch function f has radius no larger than $M + R$, and thus

$$\operatorname{Sup}\{|f'(z)|(1 - |z|^2) | z \in D\} \leqslant (M + R)/B$$

where B is Bloch's constant.

To show the converse it must be shown that, if $\mathbf{C} - E$ contains discs of arbitrarily large radii, then there exists an analytic function f on D which is not a Bloch function but satisfies the condition (∗).

It is easy to see that $\mathbf{C} - E$ must contain an infinite sequence of disjoint discs of the form $D_n = \{w \in C |\, |w - w_n| < n\}$ with $|w_{n+1}| > |w_n| + 2n + 1$ for each $n \geqslant 1$. For each n it is possible to construct a narrow open channel G_n joining D_n to D_{n+1} so that the following conditions both hold.

(i) $G = \bigcup_{n=1}^{\infty}(D_n \cup G_n)$ is simply connected.

(ii) G does not contain any disc of radius larger than 1 centered at any point of G_n.

By the Riemann mapping theorem there exists a 1-1 onto conformal map f: $D \to G$. Since f has the schlicht discs D_n in its range it cannot be a Bloch function.

Suppose $z \in D$ and $f(z) \in E$. Then $f(z) \in G_n$ for some n since $D_n \subseteq \mathbf{C} - E$ for each n. By applying the $\frac{1}{4}$ theorem [5, Theorem 14.14] to the function $s \mapsto f((s + z)/(1 + \bar{z}s))$ it follows that the range of f contains a (schlicht) disc of radius $\frac{1}{4}|f'(z)|(1 - |z|^2)$ centered at $f(z)$. Thus $\frac{1}{4}|f'(z)|(1 - |z|^2) \leqslant 1$ by (ii) and so $\operatorname{Sup}\{|f'(z)|(1 - |z|^2) | z \in f^{-1}(E)\} \leqslant 4$. The proof of the theorem is now complete.

REMARKS. (1) If $E \subseteq \mathbf{C}$ is such that the radii of the discs contained in

$\mathbf{C} - E$ are bounded and f is a meromorphic function satisfying $(*)$ then f must be analytic and thus a Bloch function (by Theorem 1).

(2) $\mathfrak{B}_0$ is defined to be the set of Bloch functions satisfying the condition that $|f'(z)|(1 - |z|^2) \to 0$ as $|z| \to 1$. Since there exist bounded functions which are not in $\mathfrak{B}_0$ (e.g. $f(z) = w_0 + \delta \exp[(z + 1)/(z - 1)]$) it follows that $E \subseteq \mathbf{C}$ satisfies

$$|f'(z)|(1 - |z|^2) \to 0 \quad \text{as } |z| \to 1 \text{ with } z \in f^{-1}(E) \Rightarrow f \in \mathfrak{B}_0 \qquad (**)$$

if and only if E is dense in $\mathbf{C}$. (Any function omitting the values E satisfies $(**)$.)

(3) In conjunction with Lappan's 5-value theorem, it can be shown that for each subset E of the Riemann sphere where E contains at least 5 points there exists an increasing function α_E: $[0, \infty) \to (0, \infty)$ such that

$$\operatorname{Sup}\left\{|f'(z)|(1 - |z|^2)/\left(1 + |f(z)|^2\right) | z \in f^{-1}(E)\right\} = M < \infty$$

implies

$$\operatorname{Sup}\left\{|f'(z)|(1 - |z|^2)/\left(1 + |f(z)|^2\right) | z \in D\right\} \leqslant \alpha_E(M).$$

The proof given by Lappan of his 5-value theorem does not give the existence of α_E. However, by applying the lemma of [**6**] to the Moebius-invariant family

$$\mathfrak{F} = \Big\{ f | f \text{ meromorphic on } D \text{ and } \operatorname{Sup}\left\{|f'(z)|(1 - |z|^2)/\left(1 + |f(z)|^2\right) | z \in f^{-1}(E)\right\} \leqslant M \Big\}$$

in the same way that Lappan applied Theorem 1 of [**4**] it follows that $\mathfrak{F}$ is a normal family. Hence, by the Marty criterion [**2**, p. 158]

$$\operatorname{Sup}\left\{|f'(0)|/\left(1 + |f(0)|^2\right) | f \in \mathfrak{F}\right\} \leqslant \alpha_E(M) < \infty.$$

Next, $f \in \mathfrak{F}$ implies $f((z + a)/(1 + \bar{a}z)) \in \mathfrak{F}$ for each $a \in D$ and this implies $|f'(a)|(1 - |a|^2)/(1 + |f(a)|^2) \leqslant \alpha_E(M)$ for each $a \in D$ and for each $f \in \mathfrak{F}$.

This minor modification of Lappan's result may be used to obtain a version of the Lappan 5-value theorem for normal meromorphic functions defined on $\mathbf{C}$.

A meromorphic function f on $\mathbf{C}$ is said to be normal if $\operatorname{Sup}\{|f'(z)|/(1 + |f(z)|^2) | z \in \mathbf{C}\} < \infty$ or equivalently if the family $\{f(e^{i\alpha}z + a) | a \in \mathbf{C}, \alpha \in [0, 2\pi)\}$ is a normal family. The maps $z \mapsto e^{i\alpha}z + a$ are the conformal isometries of $\mathbf{C}$ in the Euclidean metric. Another equivalent condition is that $\{f(z + a) | a \in \mathbf{C}\}$ be a normal family. These equivalences follow from the Marty criterion [**2**, p. 158].

THEOREM 2. *Let E be a subset of the Riemann sphere containing at least* 5 *points. Let f be a meromorphic function on* $\mathbf{C}$ *satisfying*

$$\operatorname{Sup}\left\{|f'(z)|/\left(1 + |f(z)|^2\right) | z \in f^{-1}(E)\right\} = M < \infty.$$

Then f is a normal function and in fact $\operatorname{Sup}\{|f'(z)|/(1-|f(z)|^2)|z \in \mathbf{C}\} \leqslant \alpha_E(M)$.

PROOF. Fix $z_0 \in \mathbf{C}$ and set $g(z) = f(z_0 + z)$ for $z \in D$. Then

$$|g'(z)|(1-|z|^2)/\left(1+|g(z)|^2\right) \leqslant |f'(z_0+z)|/\left(1+|f(z_0+z)|^2\right) \leqslant M$$

for $z \in g^{-1}(E)$. Thus $|g'(0)|/(1+|g(0)|^2) = |f'(z_0)|/(1+|f(z_0)|^2) \leqslant \alpha_E(M)$ as required.

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