A NECESSARY AND SUFFICIENT CONDITION FOR BLOCH FUNCTIONS

RICHARD M. TIMONEY

ABSTRACT. A necessary and sufficient condition is given for a subset \( E \subseteq \mathbb{C} \) to satisfy

\[
\text{Sup} \left\{ \left| f'(z) \right| (1 - |z|^2) \mid z \in f^{-1}(E) \right\} < \infty
\]

\[\Rightarrow \text{Sup} \left\{ \left| f'(z) \right| (1 - |z|^2) \mid z \in D \right\} < \infty\]

when \( f: D \to \mathbb{C} \) is analytic. The condition is that the complement of \( E \) should not contain large discs.

A Bloch function on \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) is an analytic function \( f: D \to \mathbb{C} \) satisfying \( \text{Sup} \{ |f'(z)| (1 - |z|^2) \mid z \in D \} < \infty \). See [1] for other characterizations and some interesting properties of Bloch functions.

The following theorem is the main result of this note.

**Theorem 1.** Suppose \( E \subseteq \mathbb{C} \). Then the condition

\[
\text{Sup} \left\{ \left| f'(z) \right| (1 - |z|^2) \mid z \in f^{-1}(E) \right\} < \infty
\]

is a sufficient condition for an analytic function \( f: D \to \mathbb{C} \) to be a Bloch function if and only if the radii of the discs contained in \( \mathbb{C} - E \) are bounded above.

For example, \( E = \mathbb{Z}^2 = \) the set of Gaussian integers works, but no finite set \( E \) will work.

This theorem was motivated by a theorem of P. Lappan [3] about normal functions.

A meromorphic function \( f \) on \( D \) is said to be normal if

\[
\text{Sup} \left\{ |f'(z)| (1 - |z|^2) / (1 + |f(z)|^2) \mid z \in D \right\} < \infty
\]

(see [1]).

**Theorem [LAPPAN 5-VALUE Theorem].** If \( f \) is a meromorphic function defined on the unit disc \( D \) and

\[
\text{Sup} \left\{ |f'(z)|(1 - |z|^2) / (1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} = M < \infty
\]

for any subset \( E \) of the Riemann sphere containing at least 5 points (3 finite points if \( f \) is analytic) then \( f \) is a normal function.

Since Bloch functions are closely related to normal functions it is natural to...
ask whether Lappan’s theorem has an analogue for Bloch functions. Theorem 1 answers this question. The author thanks L. A. Rubel for calling his attention to the problem.

**Proof (Theorem 1).** Suppose that the radii of the discs contained in \( C - E \) are bounded by \( R \) and \( f: D \rightarrow C \) is an analytic function with

\[
\sup \{|f'(z)|(1 - |z|^2)|z \in f^{-1}(E)\} = M < \infty.
\]

If \( f \) is not a Bloch function, then for each \( r > 0 \) there exists a schlicht disc \( \Delta = \{w \in C| |w - w_0| < r + R + 1\} \) in the range of \( f \) (i.e. \( f \) has a single-valued analytic inverse on \( \Delta \)—see [1]).

By hypothesis, \( \{w \in C| |w - w_0| < R + 1\} \subseteq C - E \) is false and so there exists \( w_0' \in E \) with \( |w_0' - w_0| < R + 1 \). Hence \( \{w \in C| |w - w_0| < r\} \) is a schlicht disc in the range of \( f \). Thus there exists a 1-1 conformal mapping \( \phi: D \rightarrow D \) so that \( (f \circ \phi)(z) = w_0' + rz \). Hence \( |f'(\phi(0))| |\phi'(0)| = r \). By the SchwarzLemma (12.5.3 of [5]) \( |\phi'(0)| < 1 - |\phi(0)|^2 \) and so

\[
|f'(\phi(0))|(1 - |\phi(0)|^2) > r.
\]

Observe that \( \phi(0) \in f^{-1}(E) \), which means that a choice of \( r \) with \( r > M \) will contradict the supposition at the beginning of the proof. Thus \( f \) must be a Bloch function and the “if” part of the theorem is proved.

Notice that the above argument shows that every schlicht disc in the range of the Bloch function \( f \) has radius no larger than \( M + R \), and thus

\[
\sup \{|f'(z)|(1 - |z|^2)|z \in D\} \leq (M + R)/B
\]

where \( B \) is Bloch’s constant.

To show the converse it must be shown that, if \( C - E \) contains discs of arbitrarily large radii, then there exists an analytic function \( f \) on \( D \) which is not a Bloch function but satisfies the condition (\( * \)).

It is easy to see that \( C - E \) must contain an infinite sequence of disjoint discs of the form \( D_n = \{w \in C| |w - w_n| < n\} \) with \( |w_{n+1}| > |w_n| + 2n + 1 \) for each \( n \geq 1 \). For each \( n \) it is possible to construct a narrow open channel \( G_n \) joining \( D_n \) to \( D_{n+1} \) so that the following conditions both hold.

(i) \( G = \bigcup_{n=1}^{\infty}(D_n \cup G_n) \) is simply connected.

(ii) \( G \) does not contain any disc of radius larger than 1 centered at any point of \( G_n \).

By the Riemann mapping theorem there exists a 1-1 onto conformal map \( f: D \rightarrow G \). Since \( f \) has the schlicht discs \( D_n \) in its range it cannot be a Bloch function.

Suppose \( z \in D \) and \( f(z) \in E \). Then \( f(z) \in G_n \) for some \( n \) since \( D_n \subseteq C - E \) for each \( n \). By applying the \( \frac{1}{4} \) theorem [5, Theorem 14.14] to the function \( s \rightarrow f((s + z)/(1 + zs)) \) it follows that the range of \( f \) contains a (schlicht) disc of radius \( \frac{1}{4}|f'(z)|(1 - |z|^2) \) centered at \( f(z) \). Thus \( \frac{1}{4}|f'(z)|(1 - |z|^2) < 1 \) by (ii) and so \( \sup \{|f'(z)|(1 - |z|^2)|z \in f^{-1}(E)\} < 4 \). The proof of the theorem is now complete.

**Remarks.** (1) If \( E \subseteq C \) is such that the radii of the discs contained in
C - E are bounded and f is a meromorphic function satisfying (⋆) then f
must be analytic and thus a Bloch function (by Theorem 1).

(2) \( \mathfrak{B}_0 \) is defined to be the set of Bloch functions satisfying the condition
that \( |f'(z)|(1 - |z|^2) \to 0 \) as \( |z| \to 1 \). Since there exist bounded functions
which are not in \( \mathfrak{B}_0 \) (e.g. \( f(z) = w_0 + \delta \exp[(z + 1)/(z - 1)] \)) it follows that
\( E \subseteq \mathbb{C} \) satisfies
\[
|f'(z)|(1 - |z|^2) \to 0 \quad \text{as } |z| \to 1 \text{ with } z \in f^{-1}(E) \Rightarrow f \in \mathfrak{B}_0 \quad (\Box)
\]
if and only if E is dense in \( \mathbb{C} \). (Any function omitting the values E satisfies
(\Box).)

(3) In conjunction with Lappan's 5-value theorem, it can be shown that for
each subset E of the Riemann sphere where E contains at least 5 points there
exists an increasing function \( \alpha_E : [0, \infty) \to (0, \infty) \) such that
\[
\text{Sup}\left\{ |f'(z)|(1 - |z|^2)/(1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} = M < \infty
\]
implies
\[
\text{Sup}\left\{ |f'(z)|(1 - |z|^2)/(1 + |f(z)|^2) \mid z \in D \right\} \leq \alpha_E(M).
\]
The proof given by Lappan of his 5-value theorem does not give the existence
of \( \alpha_E \). However, by applying the lemma of [6] to the Moebius-invariant family
\( \mathcal{F} = \{ f \mid f \text{ meromorphic on } D \} \)
\[
\text{Sup}\left\{ |f'(z)|(1 - |z|^2)/(1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} \leq M
\]
in the same way that Lappan applied Theorem 1 of [4] it follows that \( \mathcal{F} \) is a
normal family. Hence, by the Marty criterion [2, p. 158]
\[
\text{Sup}\left\{ |f'(0)|/(1 + |f(0)|^2) \mid f \in \mathcal{F} \right\} \leq \alpha_E(M) < \infty.
\]
Next, \( f \in \mathcal{F} \) implies \( f((z + a)/(1 + \bar{a}z)) \in \mathcal{F} \) for each \( a \in D \) and this
implies \( |f(a)|(1 - |a|^2)/(1 + |f(a)|^2) \leq \alpha_E(M) \) for each \( a \in D \) and for each
\( f \in \mathfrak{F} \).

This minor modification of Lappan's result may be used to obtain a version
of the Lappan 5-value theorem for normal meromorphic functions defined on
\( \mathbb{C} \).

A meromorphic function \( f \) on \( \mathbb{C} \) is said to be normal if \( \text{Sup}\{|f'(z)|(1 + |f(z)|^2)|z \in \mathbb{C}\} < \infty \) or equivalently if the family \( \{ f(e^{ia}z + a) \mid a \in \mathbb{C}, a \in [0, 2\pi) \} \) is a normal family. The maps \( z \mapsto e^{ia}z + a \) are the conformal
isometries of \( \mathbb{C} \) in the Euclidean metric. Another equivalent condition is that
\( \{ f(z + a) \mid a \in \mathbb{C} \} \) be a normal family. These equivalences follow from the
Marty criterion [2, p. 158].

**Theorem 2.** Let E be a subset of the Riemann sphere containing at least 5
points. Let \( f \) be a meromorphic function on \( \mathbb{C} \) satisfying
\[
\text{Sup}\left\{ |f'(z)|/(1 + |f(z)|^2) \mid z \in f^{-1}(E) \right\} = M < \infty.
\]
Then \( f \) is a normal function and in fact 
\[
\sup\{|f'(z)|/(1 - |f(z)|^2)|z \in \mathbb{C}\} < \alpha_{\mathbb{E}}(M).
\]

**Proof.** Fix \( z_0 \in \mathbb{C} \) and set \( g(z) = f(z_0 + z) \) for \( z \in D \). Then
\[
|g'(z)|(1 - |z|^2)/(1 + |g(z)|^2) < |f'(z_0 + z)|/(1 + |f(z_0 + z)|^2) < M
\]
for \( z \in g^{-1}(E) \). Thus \( |g'(0)|/(1 + |g(0)|^2) = |f'(z_0)|/(1 + |f(z_0)|^2) < \alpha_{\mathbb{E}}(M) \) as required.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

Current address: Department of Mathematics, Indiana University, Bloomington, Indiana 47401