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A COUNTEREXAMPLE TO A CONJECTURE OF A. H. STONE

HAROLD BELL AND R. F. DICKMAN, JR.

ABSTRACT. A. H. Stone has offered a sequence, $\{S(n); n > 2\}$, of conjectures characterizing multicoherence for locally connected, connected, normal spaces. The conjecture $S(n)$ is, "X is multicoherent if and only if X can be represented as the union of a circular chain of continua containing exactly n elements". It is known that $S(3)$ always obtains and that $S(6)$ obtains if the space is compact. In this paper, we construct a multicoherent plane Peano continuum C for which $S(7)$ fails. Since $S(n+1)$ implies $S(n)$, $n > 2$, $S(n)$ fails for C for all $n > 6$. Furthermore we show that for any integer $n \geqslant 3$ there exists a plane Peano continuum for which $S(2n)$ obtains while $S(2n+1)$ fails.

Introduction. Throughout this paper X will denote a locally connected, connected normal space. By a *continuum* we mean a closed and connected (not necessarily compact) subset of X. For $A \subset X$, $b_0(A)$ denotes the number of components of A less one (or ∞ if this number is infinite). The *degree of multicoherence*, $r(X)$, of X is defined by

$$r(X) = \sup\{b_0(H \cap K): X = H \cup K \text{ and } H \text{ and } K \text{ are subcontinua of } X\}.$$

If $r(X) = 0$, X is said to be *unicoherent* and we say that X is *multicoherent* otherwise. By a *chain* κ in X we mean a finite collection of subcontinua of X that can be ordered $\kappa = \{K_1, K_2, \ldots, K_n\}$ so that $K_i \cap K_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. A *circular chain* in X is a collection of subcontinua κ such that no three members of κ have a point in common and if $K \in \kappa$, then $\kappa - \{K\}$ is a chain in X. Let $n > 2$ be an integer and let $S(n)$ denote the following statement:

$S(n)$: X is multicoherent if and only if X can be represented as the union of a circular chain containing exactly n elements.

In a private communication, A. H. Stone conjectured that $S(n)$ is true for all $n > 2$ and he stated that he had established $S(n)$ for all $n > 2$ whenever $0 < r(X) < \infty$. A. D. Wallace established $S(3)$ for Peano continua in **[4]**. A. H. Stone announced $S(3)$ for locally connected normal spaces in **[3]** and in **[2]**, the second author included a proof of $S(3)$ for such spaces. In **[1]**, the second author showed that $S(4)$ obtains for a large class of spaces and in **[2]**, he showed that $S(6)$ always obtains if X is compact. The purpose of this note

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is to give an example of a multicoherent plane Peano continuum for which $S(7)$ fails.

LEMMA θ. *Let a and b be distinct points in X and suppose that $X - \{a, b\} = R \cup P \cup Q$ where R, P and Q are pairwise disjoint open connected sets and $\bar{R} \cap \bar{P} \cap \bar{Q} = \{a, b\}$. If κ is a circular chain in X and $\bigcup \kappa = X$ and some $K \in \kappa$ lies entirely in R, then $(\bar{P} \cup \bar{Q})$ meets at most four elements of κ.*

PROOF. Notice that $\{a, b\}$ is the boundary of each of P, Q and R. Let $K_0 \subset R$ and $\{K_1, K_2, \ldots, K_n\}$ be a chain representation of $\kappa - \{K_0\}$. Let S be the union of those $K \in \kappa$ that contain a, let T be the union of those $K \in \kappa$ that contain b, and let V be the union of those $K \in \kappa$ that contain neither a or b. Clearly, V has at most two components, one of which contains K_0 and is therefore contained in R. It follows that either $\bar{P}$ or $\bar{Q}$ must fail to intersect V. Therefore either $\bar{Q} \subset S \cup T$ or $\bar{P} \subset S \cup T$. Since both P and Q are connected and intersect both S and T it follows that $S \cap T \neq \varnothing$. That is, there is a $k \geqslant 0$ such that $\{a, b\} \subset K_k \cup K_{k+1}$. It follows that $K_i \subset R$ except possibly when $i = k-1, k, k+1$ or $k+2$.

Construction of the example. For each positive integer i, let C_i be a one-dimensional simplicial complex in the plane, with V_i as its set of vertices and $\mathfrak{S}_i$ as its set of edges, constructed as follows:

Let V_1 consist of three evenly distributed points on the unit circle $\{z: |z| = 1\}$ and let $\mathfrak{S}_1$ consist of three connecting open intervals. Then $C_1 = \bigcup \{I: I \in \mathfrak{S}_1\} \cup V_1$. Let $\mathfrak{U}_1 = \{U(I): I \in \mathfrak{S}_1\}$ be a set of mutually disjoint bounded open convex sets such that $I \subset U(I)$ for $I \in \mathfrak{S}_1$. Suppose C_n, V_n, $\mathfrak{U}_n$, and $\mathfrak{S}_n$ have been defined. For each $I \in \mathfrak{S}_n$ let $a(I)$, $b(I)$ be the endpoints of I, and $m(I)$ its midpoint $(a(I) + b(I))/2$. Let t_n be chosen so that $1 < t_n < 1 + 1/n$ and (for all $I \in \mathfrak{S}_n$) the two "half-open" line segments $(a(I), t_n m(I)]$, $[t_n m(I), b(I))$ are both contained in $U(I)$. Let $V_{n+1} = V_n \cup \{m(I): I \in \mathfrak{S}_n\} \cup \{t_n m(I): I \in \mathfrak{S}_n\}$. Let $\mathfrak{S}_{n+1} = \{(a(I), m(I)): I \in \mathfrak{S}_n\} \cup \{(m(I), b(I)): I \in \mathfrak{S}_n\} \cup \{(a(I), t_n m(I)): I \in \mathfrak{S}_n\} \cup \{(t_n m(I), b(I)): I \in \mathfrak{S}_n\}$. Then $C_{n+1} = \bigcup \{I: I \in \mathfrak{S}_{n+1}\} \cup V_{n+1}$. Let $\mathfrak{U}_{n+1} = \{U(I): I \in \mathfrak{S}_{n+1}\}$ be a set of mutually disjoint open convex sets such that if $I \in \mathfrak{U}_{n+1}$ and $J \in \mathfrak{U}_n$ and $I \subset U(J)$ then $I \subset U(I) \subset U(J)$. Let $D = \bigcup_{i=1}^{\infty} C_i$ and let $C = \bar{D}$.

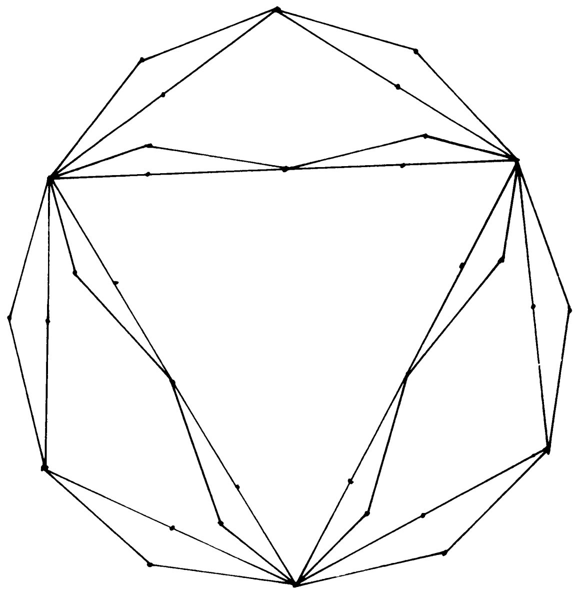
It is clear that C is multicoherent. By Theorems 3 and 6 of [2], $S(6)$ obtains for C. We will now show that $S(7)$ fails for C. (Since $(Sn + 1)$ always implies $S(n)$, $S(k)$ for $k > 6$ fails for C.)

DEFINITION. For $I = (a(I), b(I)) \in \mathfrak{S}_n$ let $I' = (a(I), t_n m(I)] \cup [t_n m(I), b(I))$; we use the convention $(I')' = I$. Let $\mathfrak{S}'_n = \{I': I \in \mathfrak{S}_n\} \cup \mathfrak{S}_n$.
The following lemma seems clear.

LEMMA A. (1) *C is a Peano continuum.*

(2) *If $I \in \mathfrak{S}'_n$ has endpoints $a(I)$ and $b(I)$ then $C - \{a(I), b(I)\}$ has exactly three components $P(I)$, $P(I')$ and $Q(I)$ where $I \subset P(I)$ and $I' \subset P(I')$.*

(3) *If, for each i, $I_i \in \mathfrak{S}'_i$ then* $\lim_{i\to\infty} \operatorname{dia}(I_i) = \lim_{i\to\infty} \operatorname{dia} P(I_i) = 0$.



C_3

Theorem 1. *The plane Peano continuum C cannot be the union of a circular chain with seven elements.*

Proof. Suppose to the contrary that κ is a circular chain with seven elements and $\bigcup \kappa = C$. Since each of the three vertices of C_1 is contained in at most two $K \in \kappa$ it follows that some $K \in \kappa$ contains no vertex of C_1 and is therefore contained in $P(I_1)$ for some $I_1 \in \mathfrak{S}_1'$. If a and b are the endpoints of I_1, we may write $C - \{a, b\}$ as the union of $R = P(I_1)$, $P = P(I_1')$ and $Q = Q(I_1)$. According to Lemma θ, $(\overline{P} \cup \overline{Q})$ meet at most four elements of κ and so R contains three elements, say K_1^1, K_2^1, and K_3^1. Now the vertex of C_2 that lies in R, must miss one of these K_i^1's, say K_1^1. Then there is an $I_2 \in \mathfrak{S}_2'$ such that $P(I_2)$ contains K_1^1. Let a_2, b_2 be the endpoints of I_2 and note that

$$C - \{a_2, b_2\} = P(I_2) \cup P(I_2') \cup Q(I_2).$$

Again by Lemma θ, $P(I_2)$ must contain 3 elements of κ and hence the new vertex of C_3 that lies in $P(I_2)$ must miss one of these elements of κ. We continue as above and select a sequence $I_2, I_3, \ldots,$ such that $I_j \in \mathfrak{S}_j'$, $P(I_{(j+1)}) \subset P(I_j)$ and each $P(I_j)$ contains a member of κ, K_j. Since κ is finite, there is a subsequence $\{I_{j_k}\}$ such that $K_{j_s} = K_{j_r}$ for all $r, s \geqslant 1$. But then $K_{j_1} \subset \bigcap_{n=2}^{\infty} P(I_n)$, and, by Lemma A, K_{j_1} must be a singleton. Of course, this is impossible and this completes the proof.

Theorem 2. *For all integers $n \geqslant 3$, there exists a plane Peano continuum $P(n)$ such that $S(2n)$ obtains but $S(2n+1)$ fails.*

Proof. We construct $P(n)$ in the same fashion we constructed C except in this instance we change the example by letting C_1 be a regular n-gon. If I is an edge in C_1 with endpoints a and b then $\overline{P(I)} \cup \overline{P(I')}$ can be easily written as the union of two continua K_I^a and K_I^b where $a \notin K_I^b$ and $b \notin K_I^a$. Then $\{K_I^a\colon I \text{ is an edge in } C_1\} \cup \{K_I^b\colon I \text{ is an edge in } C_1\}$ is a circular chain that covers $P(n)$ and has $2n$ elements. Let κ be any circular chain that covers $P(n)$. If every element of κ contains a vertex of C_1 then κ has at most $2n$ elements. If some element of κ fails to contain a vertex of C_1 then there is an $I \in \mathfrak{S}_1'$ such that $P(I)$ contains an element of κ. The same proof used for $P(3)$ then shows that in this case κ has at most 6 elements.

REMARK. The authors have not been able to construct an example of a plane Peano continuum P for which $S(2n + 1)$ obtains while $S(2n + 2)$ fails.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221

DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061