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# A FIXED POINT THEOREM AND ATTRACTORS

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ABSTRACT. We investigate attractors for compact sets by considering a certain quotient space. The following theorem is included. Let $f$: $G \to G$, $G$ a closed convex subset of a Banach space, $f$ a mapping satisfying (i) there exists $M \subset G$ which is an attractor for compact sets under $f$; (ii) the family $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. Then $f$ has a fixed point.

**1.** Since the appearance of the celebrated Schauder fixed point theorem, the following has been a longstanding open question.

CONJECTURE 1. Let $f$: $G \to G$, $G$ a closed, convex subset of a Banach space, $f$ a continuous mapping with $f^N$ compact, $N > 1$. Then $f$ has a fixed point (?)

In a paper which appeared in 1972, Roger Nussbaum **[6]** states Conjecture 1 along with several new conjectures concerning fixed points. In that paper he introduces the concept of an attractor for compact sets.

DEFINITION 1.1. Let $X$ be a topological space, $f$: $X \to X$ a map, and $M$ a nonempty subset of $X$. $M$ is an attractor for compact sets under $f$ if (1) $M$ is compact and $f(M) \subset M$, and (2) given any compact set $C \subset X$ and any open neighborhood $U$ of $M$, there exists an integer $N = N(C, U)$ such that $f^n(C) \subset U$ for $n \geqslant N$. $M$ is an attractor for points under $f$ if (2) holds for $C = \{x\} \subset X$. For brevity, we shall use "a.c.s. under $f$" for the phrase "attractor for compact sets under $f$."

In this setting of Conjecture 1, we see that $\text{cl}(f^N(G))$ is an a.c.s. under $f$; hence, an affirmative answer to the following conjecture of Nussbaum would yield an affirmative answer to Conjecture 1.

CONJECTURE 2. Let $G$ be a closed, convex subset of a Banach space $X$ and $f$: $G \to G$ a continuous map. If there exists a set $M \subset G$ which is an attractor for compact sets under $f$, then $f$ has a fixed point (?)

Nussbaum **[6]** states that he does not think that Conjecture 2 has an affirmative answer. If one attempts to provide a counterexample to Conjecture 2, a characterization of functions which have attractors seems desirable. In the direction of obtaining information concerning the structure of a.c.s., see Solomon **[7]**.

In §2, we provide several results concerning those functions which have a.c.s. In §3, we provide a partial affirmative answer to Conjecture 2 and thus

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Conjecture 1, for functions $f$ satisfying the additional condition that the sequence of iterates, $\{f^n\}$, of $f$ be equicontinuous.

**2. A quotient space.** Throughout the remainder of this paper, $X$ is assumed to be a metrizable topological space.

DEFINITION 2.1. Let $f$ be a continuous self-map on the metrizable topological space $X$ with $M$ an $f$-invariant compact subset of $X$. We denote by $X/M$ the quotient space equipped with the quotient topology arising from $X$ by identifying $M$ to a point. $f^*$: $X/M \to X/M$ will denote the induced map which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \Pi\downarrow & & \downarrow\Pi \\ X/M & \xrightarrow{f^*} & X/M \end{array}$$

commute where $\Pi$ is the natural projection.

The following facts concerning $X/M$ are easily established.

LEMMA 2.1. (1) *If $U$ is an open subset of $X$ and $M \subset U$, then $\Pi(U)$ is open in $X/M$.* (2) $\Pi$ *is a closed map and since* $\{\Pi^{-1}(x^*)\}$ *is compact for every* $x^*$ *in* $X/M$, $\Pi$ *is a perfect map.*

Employing a well-known result of Hanai [2], the image under a continuous, closed perfect map of a metrizable topological space is metrizable–we obtain the following lemma.

LEMMA 2.2. *$X/M$ is metrizable.*

LEMMA 2.3. *If $Y^*$ is a compact subset of $X/M$ such that $a^* = \Pi(M) \in Y^*$, then $\Pi^{-1}(Y^*)$ is compact.*

PROOF. The continuity of $\Pi$ asserts that $\Pi^{-1}(Y^*)$ is closed in $X$. Let $\{U_\alpha\}$ be an open covering of $\Pi^{-1}(Y^*)$. Since $M \subset \Pi^{-1}(Y^*)$ and $M$ is compact, there exist indices $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $M \subset \bigcup_{i=1}^m U_{\alpha_i}$. Let $W = \bigcup_{i=1}^m U_{\alpha_i}$ and define the open covering $\{V_\alpha\}$ by $V_\alpha = U_\alpha \cup W$. Note that $M \subset V_\alpha$ for each $\alpha$ which implies that $\Pi(V_\alpha)$ is open in $X/M$. Since $\{\Pi(V_\alpha)\}$ covers $Y^*$, there exists a finite subcovering $\Pi(V_{\beta_1}), \Pi(V_{\beta_2}), \ldots, \Pi(V_{\beta_n})$ which covers $Y^*$. Since $\Pi^{-1}(\Pi(V_\alpha)) = V_\alpha$ for each $\alpha$, this implies that $\{V_{\beta_i}\}_{i=1}^n$ covers $\Pi^{-1}(Y^*)$ which in turn implies that the collection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_2}, \ldots, U_{\alpha_m}, U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}\}$ covers $\Pi^{-1}(Y^*)$; thus, $\Pi^{-1}(Y^*)$ is compact.

LEMMA 2.4. *If $M$ is an attractor for compact sets under $f$ and $x^* \in X/M$, then the sequence $\{f^{*n}(x^*)\}$ converges to $a^*$.*

PROOF. If $x^* = a^*$ the statement is clearly true since $f^{*n}(a^*) = a^*$ for each $n$. If $x^* \neq a^*$ then $\Pi^{-1}(x^*)$ is a singleton, $\{x\}$, hence a compact set in $X$. Choosing an arbitrary open set $U^*$ in $X/M$ containing $a^*$, we obtain an open set $\Pi^{-1}(U^*)$ in $X$ containing $M$. Since $M$ is an a.c.s. under $f$, there is a positive integer $N$ such that $f^n(x) \in \Pi^{-1}(U^*)$ for each $n \geqslant N$; hence $f^{*n}(x^*) \in U^*$ for each $n \geqslant N$.

LEMMA 2.5. *If $M$ in an a.c.s. under $f$ and $Y^*$ is a nonempty, $f^*$-invariant compact subset of $X/M$, then $\bigcap_{n=1}^{\infty} f^{*n}(Y^*) = \{a^*\}$.*

PROOF. Since $f(Y^*) \subset Y^*$, Lemma 2.4 implies that $a^* \in Y^*$ and Lemma 2.3 implies that $\Pi^{-1}(Y^*)$ is compact. Let $U^*$ be any open set in $X/M$ containing $a^*$. Since $M$ is an a.c.s. under $f$ there is a positive integer $N$ such that $f^n(\Pi^{-1}(Y^*)) \subset \Pi^{-1}(U^*)$ for $n \geqslant N$, implying $f^{*n}(Y^*) \subset U^*$. Since $U^*$ was arbitrary, our lemma follows.

We now provide a statement which relates attractors in $X$ and $X/M$.

THEOREM 2.1. *Let $f$ be a continuous self-map on a metrizable topological space $X$ and $M$ a nonempty compact $f$-invariant subset of $X$. Then $M$ is an attractor for compact sets under $f$ if and only if $\{a^*\}$, with $a^* = \Pi(M)$, is an attractor for compact sets under the induced mapping $f^*$.*

PROOF. Let $M$ be an a.c.s. under $f$ and $C^*$ any compact set in $X/M$, then $C^* \cup \{a^*\}$ is also compact. If $O^*$ is any open neighborhood of $a^*$, then $\Pi^{-1}(O^*)$ is open in $X$ and contains $M$. From Lemma 2.3, $\Pi^{-1}(C^* \cup \{a^*\})$ is compact, thus there exists $N$, a positive integer, such that $f^n(\Pi^{-1}(C^* \cup \{a^*\})) \subset \Pi^{-1}(O^*)$ for each $n \geqslant N$ implying that $f^{*n}(C^* \cup \{a^*\}) \subset O^*$; hence, $\{a^*\}$ is an a.c.s. under $f^*$.

Conversely, let us assume that $\{a^*\}$ is an a.c.s. under $f^*$. Let $C$ be any compact set in $X$. If $O$ is any open set containing $M$, we consider the compact set $\Pi(C)$ and the open set $\Pi(O)$ in $X/M$ containing $\{a^*\}$. Since $\{a^*\}$ is an attractor for compact sets under $f^*$, there is a positive integer $N$ such that $f^{*n}(\Pi(C)) \subset \Pi(O)$ for $n \geqslant N$; thus, it follows that $f^n(C) \subset O$ showing that $M$ is an a.c.s. under $f$.

Let $(Y, d)$ be a metric space and $f$ a self-map on $Y$. We say that $f$ is contractive in the sense of Edelstein if for each $y_1, y_2$ in $Y$, with $y_1 \neq y_2$, we have that $d(f(y_1), f(y_2)) < d(y_1, y_2)$.

In the sequel we shall need the following result of Janos **[4]**.

THEOREM 2.2. *If $f$ is a continuous self-map on a metrizable topological space $X$ such that $\{f^n(x)\}_{n=1}^{\infty}$ converges for each $x$ in $X$, then the following statements are equivalent:*

(1) *$X$ can be metrized in such a way that $f$ becomes contractive in the sense of Edelstein;*

(2) *For a nonempty compact $f$-invariant subset $Y$ of $X$, the set $\bigcap_{n=1}^{\infty} f^n(Y)$ is a singleton.*

By employing Lemma 2.4, Lemma 2.5, and Theorem 2.2, we have the following.

THEOREM 2.3. *If $M$ is an a.c.s. under $f$: $X \to X$, then there exists a metric $d^*$,* *compatible with the topology on $X/M$, relative to which the induced map $f^*$ is contractive in the sense of Edelstein.*

To see that the converse of Theorem 2.3 is not in general true, we provide the following.

EXAMPLE 1. Let $B$ denote the closed unit disc in the real Hilbert space $l_2$. Define $f\colon B \to B$ as follows. For $x = (x_1, x_2, \dots) \in B$, $f(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots)$, where $\alpha_i = 1 - 1/(i+1)^2$, $i = 1, 2, \dots$. Since $f$ is a linear self-map and for any arbitrary but fixed nonzero $x$ in $B$ we have, letting $N$ denote the smallest integer such that $x_N \neq 0$, that

$$\begin{aligned}\|f(x)\|^2 &= \alpha_N^2|x_N|^2 + \sum_{N+1}^{\infty} \alpha_i^2|x_i|^2 \\ &\leqslant \alpha_N^2|x_N|^2 + \|x\|^2 - |x_N|^2 = \|x\|^2 - (1-\alpha_N^2)|x_N|^2\end{aligned}$$

or

$$\|f(x)\|^2 < \|x\|^2,$$

implying $\|f(x)\| < \|x\|$; i.e., $f$ is contractive. We also observe that

$$\|f^n(x)\|^2 \geqslant \left(\prod_{i=0}^{n-1} \alpha_{N+i}\right)|x_N|^2 > \frac{N}{N+1}\,|x_N|^2,$$

since

$$\prod_{i=N}^{N+k} \alpha_i = \prod_{i=1}^{N+k} \alpha_i \Big/ \prod_{i=1}^{N-1} \alpha_i = \frac{(N+k+2)N}{(N+k+1)(N+1)} > \frac{N}{N+1}, \quad \text{for } k \geqslant 0.$$

Hence, $f^n(x) \nrightarrow 0$, the unique fixed point, as $n \to \infty$. Thus, the existence of a unique fixed point $x_0$ with $f$ a contractive mapping is not sufficient to conclude that $\{x_0\}$ is an attractor for compact sets under $f$.

**3. A fixed point theorem.** In 1965, Felix Browder [**1**] and W. A. Kirk [**5**] produced, independently, a fixed point theorem for nonexpansive self-maps defined on a closed convex subset of a Banach space. Both relied heavily upon the concept of normal structure.

The conditions of the theorem which we shall present are less stringent in one respect, namely, the nonexpansiveness of $f$ is relaxed to the assumption of equicontinuity of the sequence of iterates $\{f^n\}$ and the normal structure is not assumed, but we postulate the existence of an attractor for compact sets under $f$.

REMARK 3.1. If $f\colon X \to X$ is a nonexpansive self-map on a metric space $(X, d)$, then the family of iterates $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. If $X$ is compact, the equicontinuity of $\{f^n\}$ is implied by assuming that for some $N \geqslant 1$, the self-map $f^N$ is nonexpansive.

The proof of our theorem makes use of the following fact (see [**3**]).

LEMMA 3.1. *Let $(X, d)$ be a compact metric space and $f\colon X \to X$ with $\{f^n\colon n \geqslant 1\}$ equicontinuous. Then there exists a retraction $r\colon X \to C_f$ where $C_f$*

*denotes the core of $f$; i.e., $C_f = \bigcap\{f^n(X): n \geqslant 1\}$.*

THEOREM 3.1. *Let $f$: $G \to G$, $G$ a closed convex subset of a Banach space, $f$ a mapping satisfying*

(i) *there exists $M \subset G$ which is an attractor for compact sets under $f$, and*

(ii) *the family $\{f^n\}_{n=1}^{\infty}$ is equicontinuous (relative to the metric induced by the norm). Then $f$ has a fixed point.*

PROOF. Let $Y$ denote the closed convex hull of $M$; i.e., $Y = \text{cl}[\text{co}(M)]$. Thus, $Y$ is a nonempty compact convex subset of $G$, not necessarily $f$-invariant. Denote by $X$ the closure of $\bigcup_{n=0}^{\infty} f^n(Y)$ $(f^0(x) = x)$. We shall now show that $X$ is $f$-invariant and compact. Since $f[f^n(Y)] = f^{n+1}(Y)$ for every $n \geqslant 0$, it follows that $\bigcup_{n=0}^{\infty} f^n(Y)$ is $f$-invariant. The continuity of $f$ yields that $X$ is $f$-invariant.

To prove that $X$ is compact, we show that $\bigcup_{n=0}^{\infty} f^n(Y)$ is totally bounded. Given $\varepsilon > 0$, we consider the $\varepsilon/2$ open neighborhood of $M$ which we denote by $M_{\varepsilon/2}$–if $d$ denotes the metric induced by the norm and $\alpha > 0$, $M_\alpha = \{x \in G: d(x, M) < \alpha\}$. From the definition of an a.c.s. under $f$, we have that there exists a positive integer $N_0$ such that $n \geqslant N_0$ implies that $f^n(Y) \subset M_{\varepsilon/2}$. Since $M$ is compact, it has a finite $\varepsilon/2$-mesh implying that $M_{\varepsilon/2}$ has a finite $\varepsilon$-mesh. Since $\bigcup_{n=0}^{N_0} f^n(Y)$ is compact; i.e., has a finite $\varepsilon$-mesh, this implies that $\bigcup_{n=0}^{\infty} f^n(Y)$ has a finite $\varepsilon$-mesh. Since $\varepsilon$ was arbitrary and $X$ is complete we have that $X$ is compact.

Let us denote by $C$ the core of the restriction of $f$ to $X$; i.e., $C = \bigcap_{n=0}^{\infty} f^n(X)$. It follows that $C$ is nonempty, compact, and $f$-invariant. $C$ being a compact $f$-invariant subset of $G$ and $M$ being an a.c.s. under $f$ implies that $C \subset M$.

Since $f$ is equicontinuous on $X$, we have by Lemma 3.1, that there exists a retraction $r$: $X \to C$. Recalling that $C \subset Y \subset X$, we can consider the self-mapping $f \circ r$: $Y \to Y$. Since $Y$ is compact, convex, and $f \circ r$ is continuous, the Schauder fixed point theorem ensures the existence of a fixed point for $f \circ r$. Thus there is a $y \in Y$ such that $f \circ r(y) = y$. But since $r(y) \in C$ and $C$ is $f$-invariant, we have that $y \in C$, which implies, since $r$ is the identity on $C$, that $r(y) = y$ which in turn furnishes the desired result $f(y) = y$.

REMARK. Our proof actually does not use the full strength of part (ii) of the hypothesis, but only the weaker assumption: (ii)* $M$ contains an $f$-invariant retract of $X$; i.e., there is some $C^* \subset M$ such that there is a retraction $r$: $X \to C^*$ and $f(C^*) \subset C^*$. Since $C^* \subset Y$ we again consider the self-map $f \circ r$: $Y \to Y$ and our proof goes through.

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