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BRAUER GROUPS OF LINEAR ALGEBRAIC GROUPS WITH CHARACTERS

ANDY R. MAGID

ABSTRACT. Let G be a connected linear algebraic group over an algebraically closed field of characteristic zero. Then the Brauer group of G is shown to be $C \times (\mathbf{Q}/Z)^{(n)}$ where C is finite and $n = d(d-1)/2$, with d the Z-rank of the character group of G. In particular, a linear torus of dimension d has Brauer group $(\mathbf{Q}/Z)^{(n)}$ with n as above.

In [**6**], B. Iversen calculated the Brauer group of a connected, characterless, linear algebraic group over an algebraically closed field of characteristic zero: the Brauer group is finite–in fact, it is the Schur multiplier of the fundamental group of the algebraic group [**6**, Theorem 4.1, p. 299]. In this note, we extend these calculations to an arbitrary connected linear group in characteristic zero. The main result is the determination of the Brauer group of a d-dimensional affine algebraic torus, which is shown to be $(\mathbf{Q}/Z)^{(n)}$ where $n = d(d-1)/2$. (This result is noted in [**6**, 4.8, p. 301] when $d = 2$.) We then show that if G is a connected linear algebraic group whose character group has Z-rank d, then the Brauer group of G is $C \times (\mathbf{Q}/Z)^{(n)}$ where n is as above and C is finite.

We adopt the following notational conventions: F is an algebraically closed field of characteristic zero, and $T = (F^*)^{(d)}$ is a d-dimensional affine torus over F. We use H^*_{et}, H^*_{cl}, and H^*_{gr} to denote étale, singular, and group cohomology. If G is an abelian group and m a positive integer, ${}_mG$ denotes the m-torsion in G. If X is an affine F-variety, $F[X]$ is its coordinate ring. $\text{Br}(\cdot)$ denotes Brauer group and $[\cdot]$ denotes the class in the Brauer group of an Azumaya algebra. $X(G)$ is the character group of the algebraic group G and $U(A)$ denotes the units group of the ring A. We let G_m denote $GL_1(F)$.

PROPOSITION 1. *Let A be a finite abelian group and let X be a smooth F-variety such that $H^i_{\text{et}}(X, A) = 0$ for $i = 1, 2$. Then $H^2_{\text{et}}(X \times T, A) = A^{(n)}$ where $n = d(d-1)/2$.*

PROOF. By the Lefschetz principle and smooth base change [**1**, Corollary 1.6, p. 211] we may assume $F = \mathbf{C}$. By the comparison theorem for classical and étale cohomology [**1**, Theorem 4.4, p. 74]

$$H^2_{\text{et}}(X \times T, A) = H^2_{\text{cl}}(X \times T, A).$$

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Let $V = \mathbf{C}^{(d)}$ and let $\Gamma = Z^{(d)}$. Then $T = V/\Gamma$. Let $Y = X \times V$. Then Γ operates on Y with $Y/\Gamma = X \times T$. Since Y is homotopically equivalent to X, $H^i_{\text{cl}}(Y, A) = H^i_{\text{cl}}(X, A)$, so by the comparison theorem $H^i_{\text{cl}}(Y, A) = 0$ for $i = 1, 2$. The cohomology spectral sequence of a covering then yields a spectral sequence

$$H^p_{\text{cl}}(\Gamma, H^q_{\text{cl}}(Y, A)) \Rightarrow H^n_{\text{cl}}(X \times T, A),$$

with $E^{2,0}_\infty = H^2_{\text{gr}}(\Gamma, A)$ and $E^{1,1}_\infty = E^{0,2}_\infty = 0$, so $H^2_{\text{cl}}(X \times T, A) = H^2_{\text{gr}}(\Gamma, A)$. By [**7**, p. 189], $H^2_{\text{gr}}(\Gamma, A)$ is of the desired form.

Corollary 2. *Let $\overline{G}$ be a simply connected linear algebraic group over F. Then*

$$_mH^2_{\text{et}}(\overline{G} \times T, G_m) = (Z/mZ)^{(n)}$$

where $n = d(d-1)/2$.

Proof. By [**4**, Corollary 4.4, p. 278], $\text{Pic}(\overline{G}) = 1$, and hence $\text{Pic}(\overline{G} \times T) = 1$. Then the Kummer sequence for m,

$$0 \to Z/mZ \to G_m \to G_m \to 1$$

shows that $_mH^2_{\text{et}}(\overline{G} \times T, G_m) = H^2_{\text{et}}(\overline{G} \times T, Z/mZ)$ and $_mH^2_{\text{et}}(\overline{G}, G_m) = H^2_{\text{et}}(\overline{G}, Z/mZ)$. By [**6**, Theorem 4.1, p. 299], $_mH^2_{\text{et}}(\overline{G}, G_m) = 1$. Also,

$$H^1_{\text{et}}(\overline{G}, Z/mZ) = \text{Hom}(\pi_1\overline{G}, Z/mZ) = 1$$

since $\overline{G}$ is simply connected. Proposition 1 now implies the result.

Corollary 3. *Let $\overline{G}$ be a simply connected linear algebraic group over F. Then*

$$H^2_{\text{et}}(\overline{G} \times T, G_m) = (\mathbf{Q}/Z)^{(n)}$$

where $n = d(d-1)/2$. In particular, $H^2_{\text{et}}(T, G_m) = (\mathbf{Q}/Z)^{(n)}$.

Proof. The first assertion follows from Corollary 2 and the fact that $H^2_{\text{et}}(\overline{G} \times T, G_m)$ is torsion [**5**, Proposition 1.4, p. 71]. The second is the special case where $\overline{G} = \{e\}$.

In the notation of Corollary 3, there is a canonical injection

$$\text{Br}(\overline{G} \times T) \to H^2_{\text{et}}(\overline{G} \times T, G_m)$$

[**5**, Proposition 1.4, p. 48]. We next construct Azumaya algebras to show that the injection is onto.

Definition. Let R be a domain with $1/m \in R$ and assume R contains a primitive mth root of unity w. Let a and b be units of R. Then $A^w_R(a, b)$ denotes the associative R-algebra with identity generated by two elements x and y subject to the relations: $x^m = a$, $y^m = b$, $yx = wxy$. We note that $A^w_R(a, b)$ is a free R-module of rank m^2.

Lemma 4. *Let R be a domain. Assume $1/m \in R$ and that R contains a*

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primitive mth root of unity w. Let a and b be units of R. Then $A_R^w(a, b)$ is an Azumaya R-algebra.

PROOF. Let $A = A_R^w(a, b)$, let M be a maximal ideal of R, and let $S = R/M$. Then $A/MA = A_S^w(a, b)$, where w, a, b represent the corresponding images in S. Since $1/m \in S$, w is still a primitive mth root of unity in S, and hence by [**8**, Theorem 15.1, p. 144], A/MA is a central simple S-algebra. Since A is free as R-module and A/MA is central simple over R/M for every maximal ideal M, A is an Azumaya R-algebra by [**2**, Theorem 4.7, p. 379].

LEMMA 5. *Let $R = F[t_1, t_1^{-1}, t_2, t_2^{-1}]$, and let w be a primitive mth root of unity in F. Then $A_R^w(t_1, t_2)$ has order m in* $\mathrm{Br}(R)$.

PROOF. Let $A = A_R^w(t_1, t_2)$ and let $K = F(t_1, t_2)$. Then $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is an injection by [**2**, Theorem 7.2, p. 388]. Thus it suffices to compute the order of $a = [A \otimes_R K] = [A_K^w(t_1, t_2)]$ in $\mathrm{Br}(K)$. By [**8**, Theorem 15.7, p. 149], $[A_K^w(c, b)] = 1$ if and only if c is a norm from $K(\sqrt[m]{b})$ to K, and by [**8**, Theorem 15.1, p. 144], $a^k = [A_K^w(t_1^k, t_2)]$. It follows that $a^m = 1$. Suppose $a^k = 1$ for $1 \leqslant k < m$. $K(\sqrt[m]{t_2}) = F(t_1, x)$ where $x^m = t_2$. If t_1^k is a norm from $F(t_1, x)$, there are relatively prime polynomials $p, q \in F[t_1, x]$ with

$$t_1^k = \prod_{i=0}^{m-1} p(t_1, w^i x) q(t_1, w^i x)^{-1},$$

so $t_1^k \prod q(t_1, w^i x) = \prod p(t_1, w^i x)$. Then $t_1 | p(t_1, w^i x)$ for some i, and hence for all i, so t_1^m divides $t_1^k \prod q(t_1, w^i x)$. It follows that t_1^{m-k} divides $q(t_1, w^i x)$ for some i, and hence for all i. In particular, $t_1 | p$ and $t_1 | q$, contrary to choice of p and q. Thus there is no such k and the result follows.

THEOREM 6. *Let w be a primitive mth root of unity in F, let $t_1, \ldots, t_d$ be a Z-basis of $X(T)$, and let $R = F[T]$. Then $\{[A_R^w(t_i, t_j)] | 1 \leqslant i < j \leqslant d\}$ is a Z/mZ-basis of ${}_mH^2_{\mathrm{et}}(T, G_m)$, and ${}_m\mathrm{Br}(T) = {}_mH^2_{\mathrm{et}}(T, G_m)$.*

PROOF. Let $a(i,j)$ denote the class $[A_R^w(t_i, t_j)]$ in $\mathrm{Br}(R)$. Suppose $x = \prod a(i,j)^{n(i,j)} = 1$ in $\mathrm{Br}(R)$. Now $R = F[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$. Fix i, j. Let $f\colon R \to F[t_i, t_i^{-1}, t_j, t_j^{-1}]$ be defined by $f(t_k) = 1$ if $k \neq i$ or $k \neq j$. Then

$$\mathrm{Br}(f)(x) = a(i,j)^{n(i,j)}$$

(in the obvious notation). By Lemma 5, $m | n(i,j)$. Thus the $a(i,j)$ generate a subgroup of $\mathrm{Br}(R)$ of order $md(d-1)/2$. By Corollary 2 with $\overline{G} = \{e\}$, ${}_mH^2_{\mathrm{et}}(T, G_m)$ has the same order, and the result follows.

COROLLARY 7. $\mathrm{Br}(T) = H^2_{\mathrm{et}}(T, G_m) = (\mathbf{Q}/Z)^{(n)}$ *where* $n = d(d-1)/2$.

PROOF. Apply Theorem 6 and Corollary 3.

The isomorphism of Theorem 6 has the following invariant description: fix an mth root of unity w in F. Then the bilinearity [**8**, p. 146] and skew-symmetry [**8**, p. 147] and [**8**, p. 94] of the A_R^w-construction shows that there is a group homomorphism

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$$\Lambda^2(X(T)) \otimes_Z Z/mZ \to {}_m\mathrm{Br}(T)$$

given by $(u \wedge v) \otimes k = A^w(u, v)^k$, and by Theorem 6 this is an isomorphism.

This isomorphism is natural in T. Consider an isogeny $\phi: T \to T$ given by

$$\phi(x_1, \ldots, x_d) = (x_1^{e_1}, \ldots, x_d^{e_d}).$$

Let $t_1, \ldots, t_d$ be the Z-basis of $X(T)$ given by $t_i(x_1, \ldots, x_d) = x_i$. Then the induced map

$$\phi^*: \Lambda^2(X(T)) \to \Lambda^2(X(T))$$

sends $t_i \wedge t_j$ to $e_i e_j(t_i \wedge t_j)$, and there is an exact sequence

$$0 \to \Lambda^2(X(T)) \to \Lambda^2(X(T)) \to C \to 0$$

where $C = \amalg\{Z/e_ie_jZ \mid 1 \leqslant i < j \leqslant d\}$. By standard techniques we can identify the kernel of $\phi^* \otimes Z/mZ$ with $\mathrm{Tor}_1(C, Z/mZ) = {}_mC$. Thus the kernel of $\mathrm{Br}(\phi)$ on ${}_m\mathrm{Br}(T)$ is isomorphic to ${}_mC$. We further observe that if $e_1 = \cdots = e_d = m$, then $\mathrm{Br}(\phi)$ is the zero map on ${}_m\mathrm{Br}(T)$, so every m-torsion element of $\mathrm{Br}(T)$ becomes trivial under the isogeny $T \to T$ which sends x to x^m.

The following lemma seems to be known, but for lack of a suitable reference we include a proof.

Lemma 8. *Let P be a reductive connected linear algebraic group over F. Then there is a torus T in P of dimension equal to the rank of $X(P)$ such that $P = (P, P) \times T$ as varieties.*

Proof. Let T_1 be a maximal torus of $S = (P, P)$ and let T_2 be a maximal torus of P containing T_1. There is a subtorus T of T_2 such that $T_2 = T_1 \times T$. T commutes with T_1 and T_1 is its own centralizer in S, so $S \cap T = \{e\}$. Also, $P = ST_2$, so $P = ST = S \times T$ (as varieties).

Theorem 9. *Let G be a connected linear algebraic group over F. Let P be a maximal reductive subgroup of G and let Π be the fundamental group of (P, P). Then $\mathrm{Br}(G) = W \times \Pi^{(d)} \times (Q/Z)^{(n)}$, where W is the Schur multiplier of Π, d is the rank of $X(G)$ and $n = d(d-1)/2$.*

Proof. As varieties, $G = U \times P$ where U is the unipotent radical of G. Since $F[U]$ is a polynomial ring, by [**2**, Proposition 7.7, p. 391], $\mathrm{Br}(G) = \mathrm{Br}(P)$, so we may assume $G = P$ is reductive. We note that $X(G)$ and $X(P)$ have the same rank. Let $S = (P, P)$ and write $P = S \times T$ as in Lemma 8. Let $\bar{S}$ be the simply connected covering group of S, and let $\bar{P} = \bar{S} \times T$. Since $\bar{S}/\Pi = S$, $\bar{P}$ is an étale covering space of P with group $\Pi \times 1 = \Pi$. By Corollary 3 and Theorem 6, $T \to \bar{P}$ induces an isomorphism $\mathrm{Br}(T) \xrightarrow{\sim} \mathrm{Br}(\bar{P})$. Let $p: \bar{P} \to P$ be the covering map and let $a \in \mathrm{Br}(P)$. Then by the above there is an $x \in \mathrm{Br}(T)$ such that $\mathrm{Br}(p)(ax) = 1$, so $\mathrm{Br}(P) = \mathrm{Br}(T)\mathrm{Br}(\bar{P}/P)$. To compute $\mathrm{Br}(\bar{P}/P)$, we note that $\mathrm{Pic}(\bar{P}) = \mathrm{Pic}(\bar{S}) = 1$ [**4**, Corollary 4.4, p. 278]. It follows from [**3**, Corollary 5.5, p. 17], that $\mathrm{Br}(\bar{P}/P) = H^2_{\mathrm{gr}}(\Pi, G_m(\bar{P}))$. Now

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$$G_m(\bar{P}) = U(F[\bar{P}]) = U(F[\bar{S}][t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}])$$

and by [**4**, Corollary 2.2, p. 273], $U(F[\bar{S}]) = F^*$. It follows that $G_m(T) \to G_m(\bar{P})$ is an isomorphism. Let $V = G_m(T)$. We then have a split exact sequence of Π-modules

$$1 \to F^* \to V \to Z^{(d)} \to 1.$$

Thus $\mathrm{Br}(\bar{P}/P) = H^2_{\mathrm{gr}}(\Pi, V) = H^2_{\mathrm{gr}}(\Pi, F^*) \times H^2_{\mathrm{gr}}(\Pi, Z^{(d)})$. Now $H^2_{\mathrm{gr}}(\Pi, F^*)$ is the Schur multiplier of Π and $H^2_{\mathrm{gr}}(\Pi, Z)$ is the character group of Π, and the latter is isomorphic to Π since Π is abelian. Thus $\mathrm{Br}(\bar{P}/P) = W \times \Pi^{(d)}$. Since $\mathrm{Br}(T) \cap \mathrm{Br}(\bar{P}/P) = 1$, $\mathrm{Br}(P) = W \times \Pi^{(d)} \times \mathrm{Br}(T)$, and the theorem follows from Corollary 7.

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Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019

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