BRAUER GROUPS OF LINEAR ALGEBRAIC GROUPS
WITH CHARACTERS

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Abstract. Let G be a connected linear algebraic group over an
algebraically closed field of characteristic zero. Then the Brauer group of G
is shown to be $C \times (\mathbb{Q}/\mathbb{Z})^n$ where C is finite and $n = d(d - 1)/2$, with d
the $\mathbb{Z}$-rank of the character group of G. In particular, a linear torus of
dimension $d$ has Brauer group $(\mathbb{Q}/\mathbb{Z})^n$ with n as above.

In [6], B. Iversen calculated the Brauer group of a connected, characterless,
linear algebraic group over an algebraically closed field of characteristic zero: the
Brauer group is finite—in fact, it is the Schur multiplier of the fundamental
group of the algebraic group [6, Theorem 4.1, p. 299]. In this note, we extend
these calculations to an arbitrary connected linear group in characteristic
zero. The main result is the determination of the Brauer group of a $d$-dimen-
sional affine algebraic torus, which is shown to be $(\mathbb{Q}/\mathbb{Z})^n$ where $n = d(d
- 1)/2$. (This result is noted in [6, 4.8, p. 301] when $d = 2$.) We then show
that if G is a connected linear algebraic group whose character group has
$\mathbb{Z}$-rank $d$, then the Brauer group of G is $C \times (\mathbb{Q}/\mathbb{Z})^n$ where n is as above
and C is finite.

We adopt the following notational conventions: $F$ is an algebraically closed
field of characteristic zero, and $T = (F^*)^d$ is a $d$-dimensional affine torus
over $F$. We use $H^*_\text{ét}$, $H^*_\text{sing}$, and $H^*_\text{gr}$ to denote étale, singular, and group
cohomology. If G is an abelian group and m a positive integer, $mG$ denotes
the m-torsion in G. If X is an affine $F$-variety, $F[X]$ is its coordinate ring.
$\text{Br}(\cdot)$ denotes Brauer group and $[\cdot]$ denotes the class in the Brauer group of an
Azumaya algebra. $X(G)$ is the character group of the algebraic group G and
$U(A)$ denotes the units group of the ring A. We let $G_m$ denote $GL_1(F)$.

Proposition 1. Let $A$ be a finite abelian group and let $X$ be a smooth
$F$-variety such that $H^i_{\text{ét}}(X, A) = 0$ for $i = 1, 2$. Then $H^2_{\text{ét}}(X \times T, A) = A^n$
where $n = d(d - 1)/2$.

Proof. By the Lefschetz principle and smooth base change [1, Corollary
1.6, p. 211] we may assume $F = \mathbb{C}$. By the comparison theorem for classical
and étale cohomology [1, Theorem 4.4, p. 74]

\[ H^2_{\text{ét}}(X \times T, A) = H^2_{\text{ét}}(X \times T, A). \]
Let $V = C^{(d)}$ and let $\Gamma = Z^{(d)}$. Then $T = V/\Gamma$. Let $Y = X \times V$. Then $\Gamma$ operates on $Y$ with $Y/\Gamma = X \times T$. Since $Y$ is homotopically equivalent to $X$, $H^i_{et}(Y, A) = H^i_{et}(X, A)$, so by the comparison theorem $H^i_{et}(Y, A) = 0$ for $i = 1, 2$. The cohomology spectral sequence of a covering then yields a spectral sequence

$$H^i_{et}(\Gamma, H^j_{et}(Y, A)) \Rightarrow H^{i+j}_{et}(X \times T, A),$$

with $E^{2,0}_{\infty} = H^2_{et}(\Gamma, A)$ and $E^{1,1}_{\infty} = E^{0,2}_{\infty} = 0$, so $H^2_{et}(X \times T, A) = H^2_{et}(\Gamma, A)$. By [7, p. 189], $H^2_{et}(\Gamma, A)$ is of the desired form.

**Corollary 2.** Let $\widetilde{G}$ be a simply connected linear algebraic group over $F$. Then

$$mH^2_{et}(\widetilde{G} \times T, G_m) = \left( \mathbb{Z}/m\mathbb{Z} \right)^{(n)}$$

where $n = d(d - 1)/2$.

**Proof.** By [4, Corollary 4.4, p. 278], $\text{Pic}(\widetilde{G}) = 1$, and hence $\text{Pic}(\widetilde{G} \times T) = 1$. Then the Kummer sequence for $m$,

$$0 \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow G_m \rightarrow G_m \rightarrow 1$$

shows that $mH^2_{et}(\widetilde{G} \times T, G_m) = H^2_{et}(\widetilde{G} \times T, \mathbb{Z}/m\mathbb{Z})$ and $mH^2_{et}(\widetilde{G}, G_m) = H^2_{et}(\widetilde{G}, \mathbb{Z}/m\mathbb{Z})$. By [6, Theorem 4.1, p. 299], $mH^2_{et}(\widetilde{G}, G_m) = 1$. Also,

$$H^1_{et}(\widetilde{G}, \mathbb{Z}/m\mathbb{Z}) = \text{Hom}(\pi_1 \widetilde{G}, \mathbb{Z}/m\mathbb{Z}) = 1$$

since $\widetilde{G}$ is simply connected. Proposition 1 now implies the result.

**Corollary 3.** Let $\widetilde{G}$ be a simply connected linear algebraic group over $F$. Then

$$H^2_{et}(\widetilde{G} \times T, G_m) = \left( \mathbb{Q}/\mathbb{Z} \right)^{(n)}$$

where $n = d(d - 1)/2$. In particular, $H^2_{et}(T, G_m) = \left( \mathbb{Q}/\mathbb{Z} \right)^{(n)}$.

**Proof.** The first assertion follows from Corollary 2 and the fact that $H^2_{et}(\widetilde{G} \times T, G_m)$ is torsion [5, Proposition 1.4, p. 71]. The second is the special case where $\widetilde{G} = \{ e \}$.

In the notation of Corollary 3, there is a canonical injection

$$\text{Br}(\widetilde{G} \times T) \rightarrow H^2_{et}(\widetilde{G} \times T, G_m)$$

[5, Proposition 1.4, p. 48]. We next construct Azumaya algebras to show that the injection is onto.

**Definition.** Let $R$ be a domain with $1/m \in R$ and assume $R$ contains a primitive $m$th root of unity $w$. Let $a$ and $b$ be units of $R$. Then $A^w_R(a, b)$ denotes the associative $R$-algebra with identity generated by two elements $x$ and $y$ subject to the relations: $x^m = a$, $y^m = b$, $yx = wxy$. We note that $A^w_R(a, b)$ is a free $R$-module of rank $m^2$.

**Lemma 4.** Let $R$ be a domain. Assume $1/m \in R$ and that $R$ contains a
primitive mth root of unity \( w \). Let \( a \) and \( b \) be units of \( R \). Then \( A^w_R(a, b) \) is an Azumaya \( R \)-algebra.

**Proof.** Let \( A = A^w_R(a, b) \), let \( M \) be a maximal ideal of \( R \), and let \( S = R/M \). Then \( A/MA = A^w_S(a, b) \), where \( w, a, b \) represent the corresponding images in \( S \). Since \( 1/m \in S \), \( w \) is still a primitive mth root of unity in \( S \), and hence by [8, Theorem 15.1, p. 144], \( A/MA \) is a central simple \( S \)-algebra. Since \( A \) is free as \( R \)-module and \( A/MA \) is central simple over \( R/M \) for every maximal ideal \( M \), \( A \) is an Azumaya \( R \)-algebra by [2, Theorem 4.7, p. 379].

**Lemma 5.** Let \( R = F[t_1, t_1^{-1}, t_2, t_2^{-1}] \), and let \( w \) be a primitive mth root of unity in \( F \). Then \( A^w_R(t_1, t_2) \) has order \( m \) in \( Br(R) \).

**Proof.** Let \( A = A^w_R(t_1, t_2) \) and let \( K = F(t_1, t_2) \). Then \( Br(R) \rightarrow Br(K) \) is an injection by [2, Theorem 7.2, p. 388]. Thus it suffices to compute the order of \( a = [A \otimes_R K] = [A^w_R(t_1, t_2)] \) in \( Br(K) \). By [8, Theorem 15.6, p. 149], \( [A^w_R(c, b)] = 1 \) if and only if \( c \) is a norm from \( K(\sqrt[m]{b}) \) to \( K \), and by [8, Theorem 15.1, p. 144], \( a^k = [A^w_R(t_1^k, t_2)] \). It follows that \( a^m = 1 \). Suppose \( a^k = 1 \) for \( 1 < k < m \). \( K(\sqrt[m]{t_2}) = F(t_1, x) \) where \( x^m = t_2 \). If \( t_1^k \) is a norm from \( F(t_1, x) \), there are relatively prime polynomials \( p, q \in F[t_1, x] \) with

\[
t_1^k p(t_1, w^i x)q(t_1, w^j x)^{-1},
\]

so \( t_1^k \parallel p(t_1, w^i x) = \prod_{i=0}^{m-1} p(t_1, w^i x) \). Then \( t_1 \parallel p(t_1, w^i x) \) for some \( i \), and hence for all \( i \), so \( t_1^m \) divides \( t_1^k q(t_1, w^i x) \). It follows that \( t_1^m \parallel q(t_1, w^i x) \) for some \( i \), and hence for all \( i \). In particular, \( t_1 \parallel p \) and \( t_1 \parallel q \), contrary to choice of \( p \) and \( q \). Thus there is no such \( k \) and the result follows.

**Theorem 6.** Let \( w \) be a primitive mth root of unity in \( F \), let \( t_1, \ldots, t_d \) be a \( Z \)-basis of \( X(T) \), and let \( R = F[T] \). Then \( \{[A^w_R(t_i, t_j)]1 < i < j < d\} \) is a \( Z/mZ \)-basis of \( mH^2_{et}(T, G_m) \), and \( mBr(T) = mH^2_{et}(T, G_m) \).

**Proof.** Let \( a(i, j) \) denote the class \( [A^w_R(t_i, t_j)] \) in \( Br(R) \). Suppose \( x = \prod a(i, j)^{n(i,j)} = 1 \) in \( Br(R) \). Now \( R = F[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}] \). Fix \( i, j \). Let \( f: R \rightarrow F[t_i, t_i^{-1}, t_j, t_j^{-1}] \) be defined by \( f(t_k) = 1 \) if \( k \neq i \) or \( k \neq j \). Then

\[
Br(f)(x) = a(i, j)^{n(i,j)}
\]

(in the obvious notation). By Lemma 5, \( m|n(i, j) \). Thus the \( a(i, j) \) generate a subgroup of \( Br(R) \) of order \( md(d - 1)/2 \). By Corollary 2 with \( G = \{e\} \), \( mH^2_{et}(T, G_m) \) has the same order, and the result follows.

**Corollary 7.** \( Br(T) = H^2_{et}(T, G_m) = (\mathbb{Q}/\mathbb{Z})^n \) where \( n = d(d - 1)/2 \).

**Proof.** Apply Theorem 6 and Corollary 3.

The isomorphism of Theorem 6 has the following invariant description: fix an mth root of unity \( w \) in \( F \). Then the bilinearity [8, p. 146] and skew-symmetry [8, p. 147] and [8, p. 94] of the \( A^w \)-construction shows that there is a group homomorphism
\[ \Lambda^2(X(T)) \otimes \mathbb{Z}/m\mathbb{Z} \to m\Br(T) \]
given by \((u \wedge v) \otimes k = A^w(u, v)^k\), and by Theorem 6 this is an isomorphism.
This isomorphism is natural in \(T\). Consider an isogeny \(\phi: T \to T\) given by
\[ \phi(x_1, \ldots, x_d) = (x_1^\alpha, \ldots, x_d^\alpha). \]
Let \(t_1, \ldots, t_d\) be the \(\mathbb{Z}\)-basis of \(X(T)\) given by \(t_i(x_1, \ldots, x_d) = x_i\). Then the induced map
\[ \phi^*: \Lambda^2(X(T)) \to \Lambda^2(X(T)) \]
sends \(t_i \wedge t_j\) to \(e_i e_j (t_i \wedge t_j)\), and there is an exact sequence
\[ 0 \to \Lambda^2(X(T)) \to \Lambda^2(X(T)) \to C \to 0 \]
where \(C = \prod \{Z/e_i e_j Z|1 < i < j < d\}\). By standard techniques we can identify the kernel of \(\phi^* \otimes \mathbb{Z}/m\mathbb{Z}\) with \(\text{Tor}_1(C, \mathbb{Z}/m\mathbb{Z}) = mC\). Thus the kernel of \(\Br(\phi)\) on \(m\Br(T)\) is isomorphic to \(mC\). We further observe that if \(e_1 = \cdots = e_d = m\), then \(\Br(\phi)\) is the zero map on \(m\Br(T)\), so every \(m\)-torsion element of \(\Br(T)\) becomes trivial under the isogeny \(T \to T\) which sends \(x\) to \(x^m\).

The following lemma seems to be known, but for lack of a suitable reference we include a proof.

**Lemma 8.** Let \(P\) be a reductive connected linear algebraic group over \(F\). Then there is a torus \(T\) in \(P\) of dimension equal to the rank of \(X(P)\) such that \(P = (P, P) \times T\) as varieties.

**Proof.** Let \(T_1\) be a maximal torus of \(S = (P, P)\) and let \(T_2\) be a maximal torus of \(P\) containing \(T_1\). There is a subtorus \(T\) of \(T_2\) such that \(T_2 = T_1 \times T\). \(T\) commutes with \(T_1\) and \(T_1\) is its own centraliser in \(S\), so \(S \cap T = \{e\}\). Also, \(P = ST_2\), so \(P = ST = S \times T\) (as varieties).

**Theorem 9.** Let \(G\) be a connected linear algebraic group over \(F\). Let \(P\) be a maximal reductive subgroup of \(G\) and let \(\Pi\) be the fundamental group of \((P, P)\). Then \(\Br(G) = W \times \Pi^{(d)} \times (Q/Z)^{(n)}\), where \(W\) is the Schur multiplier of \(\Pi\), \(d\) is the rank of \(X(G)\) and \(n = d(d - 1)/2\).

**Proof.** As varieties, \(G = U \times P\) where \(U\) is the unipotent radical of \(G\). Since \(F[U]\) is a polynomial ring, by [2, Proposition 7.7, p. 391], \(\Br(G) = \Br(P)\), so we may assume \(G = P\) is reductive. We note that \(X(G)\) and \(X(P)\) have the same rank. Let \(S = (P, P)\) and write \(P = S \times T\) as in Lemma 8. Let \(\widetilde{S}\) be the simply connected covering group of \(S\), and let \(\widetilde{P} = \widetilde{S} \times T\). Since \(\widetilde{S}/\Pi = \widetilde{S}, \widetilde{P}\) is an étale covering space of \(P\) with group \(\Pi \times 1 = \Pi\). By Corollary 3 and Theorem 6, \(T \to \widetilde{P}\) induces an isomorphism \(\Br(T) \iso \Br(\widetilde{P})\). Let \(p: \widetilde{P} \to P\) be the covering map and let \(a \in \Br(P)\). Then by the above there is an \(x \in \Br(T)\) such that \(\Br(p)(ax) = 1\), so \(\Br(P) = \Br(T)\Br(\widetilde{P}/P)\). To compute \(\Br(\widetilde{P}/P)\), we note that \(\text{Pic}(\widetilde{P}) = \text{Pic}(\widetilde{S}) = 1\) [4, Corollary 4.4, p. 278]. It follows from [3, Corollary 5.5, p. 17], that \(\Br(\widetilde{P}/P) = H^2_g(\Pi, G_m(P))\). Now
\[ G_m(\bar{P}) = U(F[\bar{P}]) = U \left( F[S][t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}] \right) \]

and by [4, Corollary 2.2, p. 273], \( U(F[S]) = F^* \). It follows that \( G_m(T) \to G_m(\bar{P}) \) is an isomorphism. Let \( V = G_m(T) \). We then have a split exact sequence of \( \Pi \)-modules

\[ 1 \to F^* \to V \to Z^{(d)} \to 1. \]

Thus \( Br(\bar{P}/P) = H^2_{Gr}(\Pi, V) = H^2_{Gr}(\Pi, F^*) \times H^2_{Gr}(\Pi, Z^{(d)}) \). Now \( H^2_{Gr}(\Pi, F^*) \) is the Schur multiplier of \( \Pi \) and \( H^2_{Gr}(\Pi, Z) \) is the character group of \( \Pi \), and the latter is isomorphic to \( \Pi \) since \( \Pi \) is abelian. Thus \( Br(\bar{P}/P) = W \times \Pi^{(d)}. \)

Since \( Br(T) \cap Br(\bar{P}/P) = 1 \), \( Br(P) = W \times \Pi^{(d)} \times Br(T) \), and the theorem follows from Corollary 7.

References


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