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COMPUTABLE ISOMORPHISM INVARIANTS FOR THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF A PLANE PROJECTIVE CURVE

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ABSTRACT. The aim of this paper is to attach computable isomorphism invariants to the fundamental groups $\pi_1(\mathbf{P}^2 - c)$ where c is an irreducible plane projective curve. We use these invariants to distinguish certain of these groups. The vehicle used to obtain these invariants is the free differential calculus of R. Fox.

I. Introduction. As we shall see below, one can find a presentation for the group $\pi_1(\mathbf{P}^2 - c)$. A presentation may not, however, tell us much about the group. In fact, it is an algorithmically unsolvable problem to decide whether two given presentations present isomorphic groups or not (cf. Rabin **[5]**). In view of this fact, it is useful to have computable isomorphism invariants for these groups. We will obtain these invariants below but, first, we need a presentation for $\pi_1(\mathbf{P}^2 - c)$.

Let c be an algebraic curve of degree d in $\mathbf{P}^2$ (complex projective space). We want a presentation for the group $\pi_1(\mathbf{P}^2 - c, e_0)$ where e_0 is any basepoint in $\mathbf{P}^2 - c$. We briefly describe, following Cheniot **[2]**, such a presentation.

There are only finitely many lines through e_0, namely those through singularities and tangent lines, which cut c in fewer than d points. Let L_g be a line through e_0 which cuts c in d distinct points, and let p be a point of L_g different from e_0 and not on c. Let $L_0, L_1, \ldots, L_m$ be lines from p which include the finitely many lines which cut c in fewer than d points. We also require $L_i \neq L_g$, $i = 0, 1, \ldots, m$. We let b be a line through e_0 different from L_g. We set $A_i = b \cap L_i$, $0 \leqslant i \leqslant m$, and $F_i = L_g \cap c$, $1 \leqslant i \leqslant d$.

THEOREM 1. *There are choices of loops η_j in b, $0 \leqslant j \leqslant m$, and γ_i, $1 \leqslant i \leqslant d$, in L_g such that η_j (resp. γ_i) surrounds A_j (resp. F_i) respectively and exclusively and that the group $\pi_1(\mathbf{P}^2 - c, e_0)$ can be presented as*

$$\pi_1(\mathbf{P}^2 - c, e_0) = \langle g_1, \ldots, g_d; g_i = \phi_{ij}, g_1 g_2 \cdots g_d = 1 \rangle, \qquad 1 \leqslant i \leqslant d, 0 \leqslant i \leqslant m,$$

where g_i is represented by γ_i and ϕ_{ij} is a word in the generators g_i which is

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represented by a loop which is obtained from γ_i by a continuous deformation in the plane L_g while the basepoint $L_g \cap b$ travels through η_j.

(For a proof, see Cheniot [2].)

We require the following results to make our algebraic apparatus work.

THEOREM 2. *If c is irreducible of degree d, then any two generators of $\pi_1(\mathbf{P}^2 - c, e_0)$ are conjugate.*

(For a proof too long to be presented here, see Theorem 3.3 of Arnold [1].)

THEOREM 3. *If c is irreducible of degree d, then $H_1(\mathbf{P}^2 - c) \cong \mathbf{Z}/d\mathbf{Z}$.*

PROOF. Abelianize the above presentation. For details, see Theorem 3.5 of Arnold [1].

Theorems 2 and 3 as given here are Theorems 2 and 3 of Zariski [6].

II. Free differential calculus. In this section we describe an algebraic apparatus which produces isomorphism invariants for presentations of the groups $\pi_1(\mathbf{P}^2 - c)$ where c is an irreducible plane projective curve. The approach taken here is a modification of the one developed in Crowell and Fox [3] and Fox [4].

Let $x_1, x_2, \ldots, x_n, \ldots$ be a set of letters and let $F[x] = F[x_1, \ldots, x_n]$ be the free group on $x_1, \ldots, x_n$. We let $\pi = \langle x_1, \ldots, x_n;\ R_1, \ldots, R_k \rangle$ be a presentation of $F[x]/\langle R_1, \ldots, R_k \rangle$ where $R_1, \ldots, R_k \in F[x]$ and $\langle \cdot \rangle$ denotes normal closure.

We denote by $\mathbf{Z}(G)$ the group ring over the integers of the group G. Given a group homomorphism $f\colon G \to H$ we can extend f linearly to get a ring homomorphism (still denoted by f) $f\colon \mathbf{Z}(G) \to \mathbf{Z}(H)$. If $G \cong F[x]/\langle R_1, \ldots, R_m \rangle$ we let $\gamma\colon F[x] \to G$ be the natural projection and $\alpha\colon G \to G/G^1$ the abelianizer where G^1 is the commutator subgroup of G. Henceforth we write $G/G^1 = H$ and $F[x]/\langle R_1, \ldots, R_m \rangle = F[x]/\langle R \rangle$.

We define a derivation of type (m, p) with $m, p \in \mathbf{Z}$ to be the linear extension of a multivalued function D from $F[x]$ to $\mathbf{Z}(H)$ which satisfies:

D_{I}: the set of values of $D(w_1 \cdot w_2) =$ the set of values of $w_1^p D(w_2) + D(w_1) w_2^m$.

D_{II}: the values of $u^{-p} D(u) u^{-m}$; $p, m \in \mathbf{Z}$, are values of $D(u^{-1})$.

REMARKS. The function D can be multivalued. It can happen that $w_1 \cdot w_2 = w_1 \cdot x_i^{-\varepsilon} \cdot x_i^{\varepsilon} \cdot w_2$, $\varepsilon = \pm 1$, while $D(w_1 \cdot w_2) \neq D[(w_1 x_i^{-\varepsilon}) \cdot (x_i^{\varepsilon} w_2)]$ computed by the product rule. A short computation shows, however, that the map $w \mapsto \alpha\gamma D(w)$ is single valued. Since we are only interested in the values $\alpha\gamma D(w)$ and since we may simply reduce a word such as $w_1 x_i^{-\varepsilon} x_i^{\varepsilon} w_2$ to $w_1 \cdot w_2$ we never encounter multiple values for D. The general multiple valuedness of D does force us to state D_{II} as a condition rather than derive it from D_{I}.

The derivations we shall need are described in the following lemma.

LEMMA 4. *For each x_j, $1 \leqslant j \leqslant n$, there is a unique derivation D_j satisfying D_{I} and D_{II} for fixed p and m, such that*

(a)

$$D_j(x_i) = \delta_{ij} = 0 \quad \textit{if } i \neq j,$$
$$= 1 \quad \textit{if } i = j,$$

(b) $\alpha\gamma D$ *is single valued.*

PROOF. It is not hard to see that there is a unique derivation D_j with $D_j(x_i) = \delta_{ij}$ which satisfies D_{I}. It follows from the rule D_{I} that $D(1) = 0$. Using this we see that $D(u \cdot u^{-1}) = 0$ which implies D_{II}. A computation shows that $\alpha\gamma D(w_1 w_2) = \alpha\gamma D(w_1 x_j^{-\varepsilon} x_j^{\varepsilon} w_2)$ from which (b) follows.

Let $\pi = \langle x_1, \ldots, x_n; R_1, \ldots, R_m\rangle$ be a presentation for G. The Alexander matrix $A(\pi)$ for the presentation π is the $n \times m$ matrix whose (i, j)th entry is $\alpha\gamma D_j(x_i) \in \mathbf{Z}(H)$. Here γ: $\mathbf{Z}(F[x]/\langle R\rangle) \to \mathbf{Z}(G)$ and α: $\mathbf{Z}(G) \to \mathbf{Z}(H)$. Since $\mathbf{Z}(H)$ is commutative we can define the following elementary ideals of π.

$\mathcal{E}_k(\pi)$ = the ideal in $\mathbf{Z}(H)$ generated by the determinants of the $(n-k) \times (n-k)$ minors of $A(\pi)$ if $0 < n - k \leqslant m$,
$= 0$ if $n - k > m$,
$= \mathbf{Z}(H)$ if $n - k \leqslant 0$.

The following theorem shows that the elementary ideals are isomorphism invariants.

THEOREM 5. *Let π and ψ be presentations of the same group. Then $\mathcal{E}^k(\pi) = \mathcal{E}^k(\psi)$ for all k.*

PROOF. The proof which is a slight generalization of a similar theorem of R. Fox can be found in Theorem 4.1 of Arnold [**1**].

In order to apply the free differential calculus to the case $G = \pi_1(\mathbf{P}^2 - c)$ the group ring $\mathbf{Z}(H)$ should be a reasonable ring in which to work and the Alexander matrix should be effectively computable from the matrix of derivatives $[D_j(R_i)]$ (with R_i reduced to avoid multiple values). First of all, when $G = \pi_1(\mathbf{P}^2 - c)$, H is the cyclic group $\mathbf{Z}/d\mathbf{Z}$. Hence the group ring $\mathbf{Z}(H)$ is the polynomial ring $\mathbf{Z}[t]$ reduced modulo d. The desirable condition on the Alexander matrix is guaranteed by the following lemma.

LEMMA 6. *Let c be an irreducible plane projective curve. The homomorphism $\alpha \circ \gamma$: $\mathbf{Z}(F[x]) \to \mathbf{Z}(H)$ carries each generator g_i, $1 \leqslant i \leqslant d$, of $\pi_1(\mathbf{P}^2 - c)$ to a single generator t of $H = H_1(\mathbf{P}^2 - c)$.*

PROOF. Let g_0 be any fixed generator of $\pi_1(\mathbf{P}^2 - c)$ and let g be any other generator. According to Theorem 2, g and g_0 are conjugates. Hence $g_0 = hgh^{-1}$ for some $h \in \pi_1(\mathbf{P}^2 - c)$. We can therefore write

$$\alpha \circ \gamma(g) = \alpha \circ \gamma\left(h g_0^{-1} h\right) = \alpha \circ \gamma(h^{-1})\alpha \circ \gamma(g_0)\alpha \circ \gamma(h) = \alpha \circ \gamma(g_0).$$

We choose $t = \alpha \circ \gamma(g_0)$.

We can now state a necessary condition for $\pi_1(\mathbf{P}^2 - c)$ to be abelian.

THEOREM 7. *Let c be an irreducible plane projective curve of degree d and let D_t be as described in Lemma 4. In order for $\pi_1(\mathbf{P}^2 - c)$ to be abelian it is* *necessary that*

$\mathcal{E}_k(c) = \mathbf{Z}(\mathbf{Z}/d\mathbf{Z})$ *for* $k \geqslant 1$ *and*

$\mathcal{E}_0(c) = (D_t(t^d)) \subset \mathbf{Z}(\mathbf{Z}/d\mathbf{Z})$.

PROOF. If $\pi_1(\mathbf{P}^2 - c)$ is abelian then it is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ by Theorem 3. Since $\mathcal{E}_k(c)$ are isomorphism invariants, it suffices to show the result for $\pi = \langle t;\ t^d \rangle$. The result now follows from the definition of the elementary ideals.

III. Examples and results. We let $A^{(n,m)}$ and $\mathcal{E}_k^{(n,m)}$ denote the Alexander matrix and elementary ideals corresponding to the derivations $D_j^{(n,m)}$ which satisfy:

$D_j^{(n,m)}(u \cdot v) = u^n D_j^{(n,m)}(v) + D^{(n,m)}(u) v^m$ and

$D_j^{(n,m)}(u^{-1}) = -u^{-n} D_j^{(n,m)}(u) \cdot u^{-m}$.

EXAMPLE 1. We let c_1 be an irreducible sextic with six cusps on a conic. In this case it can be shown (Zariski [6]) that $\pi_1(\mathbf{P}^2 - c_1) = \langle g_1, g_2;\ g_1^2, g_2^3 \rangle$. We let c_2 be an irreducible nonsingular sextic. A theorem of Zariski says that $\pi_1(\mathbf{P}^2 - c_2) \cong \mathbf{Z}/6\mathbf{Z}$. The two groups $\pi_1(\mathbf{P}^2 - c_1)$ and $\pi_1(\mathbf{P}^2 - c_2)$ are distinguishable by the elementary ideals.

THEOREM 8. *Let* c_1 *and* c_2 *be as above. The elementary ideal* $\mathcal{E}_0(1, 2)$ *distinguishes* $\pi_1(\mathbf{P}^2 - c_1)$ *and* $\pi_1(\mathbf{P}^2 - c_2)$.

PROOF. Since we are interested in the values $\alpha\gamma D(w)$ and $\alpha\gamma D$ is single valued for a derivation D, we may use the product rule $D(uv) = uDv + Du \cdot v^2$ and $D(u^{-1}) = -u^{-1}D(u)u^{-2}$. We get

$$D_1(g_1^2) = g_1 + g_1^2, \quad D_2(g_2^3) = g_2(g_2 + g_2^1) + g_2^4 = g_2^2 + g_2^3 + g_2^4,$$

$$A^{(1,2)}(c_1) = \begin{pmatrix} t + t^2 & 0 \\ 0 & t^2 + t^3 + t^4 \end{pmatrix},$$

$$\mathcal{E}_0^{(1,2)}(c_1) = (1 + t^3 + 2t^4 + 2t^5),$$

$$\mathcal{E}_1^{(1,2)}(c_1) = (t + t^2, t^2 + t^3 + t^4) = \mathbf{Z}(\mathbf{Z}/6\mathbf{Z}),$$

$$\mathcal{E}_k^{(1,2)}(c_1) = \mathbf{Z}(\mathbf{Z}/6\mathbf{Z}), \qquad k > 1.$$

For the curve c_2 with $\pi_1(\mathbf{P}^2 - c_2) = \langle g;\ g^6 \rangle$

$$\mathcal{E}_0^{(1,2)}(c_2) = (1 + t + t^2 + t^3 + t^4 + t^5),$$

$$\mathcal{E}_k^{(1,2)}(c_2) = \mathbf{Z}(\mathbf{Z}/6\mathbf{Z}), \qquad k \geqslant 1.$$

In view of the above computation it suffices to show that $1 + t^3 + 2t^4 + t^5$ is not divisible by $1 + t + t^2 + t^3 + t^4 + t^5$ in the ring $\mathbf{Z}(\mathbf{Z}/6\mathbf{Z})$. For this, suppose that there are integers a, b, c, d, e and f such that

$$(a + bt + ct^2 + dt^3 + et^4 + ft^5) \cdot (1 + t + t^2 + t^3 + t^4 + t^5) = 1 + t^3 + 2t^4 + 2t^5.$$

This implies the contradictory statements $a + b + c + d + e + f = 0$ and $a + b + c + d + e + f = 1$. The last assertion of the theorem is clear since if

$\pi_1(\mathbf{P}^2 - c_1)$ were abelian we would have $\pi_1(\mathbf{P}^2 - c_1) \cong H_1(\mathbf{P}^2 - c_1) \cong \mathbf{Z}(\mathbf{Z}/6\mathbf{Z}) \cong \pi_1(\mathbf{P}^2 - c_2)$.

REMARK. Computations show that $\mathcal{E}_k^{(1,0)}$, $\mathcal{E}_k^{(1,1)}$ and $\mathcal{E}_k^{(-1,1)}$ fail to distinguish the groups $\pi_1(\mathbf{P}^2 - c_1)$ and $\pi_1(\mathbf{P}^2 - c_2)$.

EXAMPLE 2. c_1 is an irreducible three cuspidal quartic and c_2 is an irreducible nonsingular quartic. In this case $H = \mathbf{Z}/4\,\mathbf{Z}$ and it can be shown that $\pi_1(\mathbf{P}^2 - c_1) = \langle g_1, g_2; g_1^2 g_2^{-2}, g_1^4, (g_1 g_2)^3 g_1^{-2} \rangle$ (cf. Zariski [6]). We have the following result.

THEOREM 9. *There exists projective plane curves of the same degree, namely the three cuspidal quartic and the nonsingular quartic, which cannot be distinguished by* $\mathcal{E}_k^{(n,m)}$ *for any integers* n, m *and* k.

PROOF. Since $H = \mathbf{Z}/'\mathbf{Z}$, and $\mathbf{Z}(H)$ is commutative, we get the following equalities.

$$\alpha\gamma D^{(1,0)}(w) = \alpha\gamma D^{(0,1)}(w) = \alpha\gamma D^{(0,3)}(w) = \alpha\gamma D^{(3,0)}(w),$$
$$\alpha\gamma D^{(1,-1)}(w) = \alpha\gamma D^{(-1,1)}(w) = \alpha\gamma D^{(1,3)}(w) = \alpha\gamma D^{(3,1)}(w),$$
$$\alpha\gamma D^{(1,2)}(w) = \alpha\gamma D^{(2,1)}(w) = \alpha\gamma D^{(2,3)}(w) = \alpha\gamma D^{(3,2)}(w) \quad \text{for all } w \in F[g_1, g_2].$$

Hence it is enough to show that the three cuspidal quartic and the nonsingular quartic cannot be distinguished by $\mathcal{E}_k^{(1,0)}$, $\mathcal{E}_k^{(1,-1)}$ and $\mathcal{E}_k^{(1,2)}$. An inspection of the list of elementary ideals below shows that we need only verify that $(t^3 + t) = (4t^2 + 4t, 6t^2 + 6, 3 + t + 3t^2 + t^3)$ and $\mathbf{Z}(\mathbf{Z}/4\mathbf{Z}) = (1 + t, 1 + t + t^2 + t^3, 1 - t + t^2, 2t + t^3)$. These facts follow from straightforward computations.

List of elementary ideals.

$$\begin{aligned}
\mathcal{E}_k^{(1,0)}(c_1) &= (1 + t + t^2 + t^3), \quad k = 0,\\
&= (1 + t, 1 + t + t^2 + t^3, 1 - t + t^2, 2t + t^3), \quad k = 1,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k > 1;\\
\mathcal{E}_k^{(-1,1)}(c_1) &= (4t^2 + 4t, 6t^2 + 6, 3 + t + 3t^2 + t^3), \quad k = 0,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k \geqslant 1;\\
\mathcal{E}_k^{(1,2)}(c_1) &= (1 + t + t^2 + t^3), \quad k = 0,\\
&= (1 + t, 1 + t + t^2 + t^3, 1 - t + t^2, 2t + t^3), \quad k = 1,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k > 1;\\
\mathcal{E}_k^{(1,0)}(c_2) &= (1 + t + t^2 + t^3), \quad k = 0,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k \geqslant 1;\\
\mathcal{E}_k^{(1,-1)}(c_2) &= (t^3 + t), \quad k = 0,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k \geqslant 1;\\
\mathcal{E}_k^{(1,2)}(c_2) &= (1 + t + t^2 + t^3), \quad k = 0,\\
&= \mathbf{Z}(\mathbf{Z}/4\mathbf{Z}), \quad k \geqslant 1.
\end{aligned}$$

REMARKS. The above procedure can be applied to presentations of the local groups $\pi_1(\bar{\Delta}_\varepsilon(0) \cap C^2 - c)$ where $\bar{\Delta}_\varepsilon(0)$ is a small closed polydisc about $0 \in C^2$, and c is an algebraic curve with a singularity at 0. We mention two of these results. (For details, see Arnold [1].)

1. There exist algebraic singularities with the same multiplicities whose local groups are indistinguishable by $\mathfrak{S}_k^{(n,m)}$ for all n, m and k. Examples are the simple cusp and the cubo-quadratic cusp. ($x = t^3, y = t^5$ near the origin.)

2. The local groups $\pi_1(\bar{\Delta}_\varepsilon(0) \cap C^2 - c)$ of the cubical cusp and the cubo-quadratic cusp are not isomorphic. They are distinguished by $\mathfrak{S}_1^{(1,0)}$.

We also remark that the coarseness of the elementary ideals as reflected in Theorem 9 and Remark 2 above is not surprising in view of the coarseness of the elementary ideals for knots in R^3. For more discussion on the coarseness of the elementary ideals, see Arnold [1].

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