PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 2, September 1978

THE LEBESGUE DECOMPOSITION THEOREM FOR PARTIALLY ORDERED SEMIGROUP-VALUED MEASURES

PANAIOTIS K. PAVLAKOS

ABSTRACT. The present paper is concerned with partially ordered semigroup-valued measures. Below are given generalizations of the classical Lebesgue Decomposition Theorem.

These results can be applied to Stone or W^* algebra-valued positive measures (cf. [3], [12], [13], [14]).

1. Preliminaries. By a partially ordered semigroup X we mean a commutative semigroup with identity 0, equipped by a partial ordering $\leqslant$, compatible with the structure of X under the conditions:

(i) If x, y, z are elements of X with $x < y$ ($x \leqslant y$ and $x \neq y$) then $x + z < y + z$.

(ii) $x + \sup E = \sup (x + E)$, whenever there exist $\sup E$ (the supremum of E in X) and $\sup(x + E)$, $E \subseteq X$, $x \in X$.

Now X is monotone complete if every majorised increasing directed family in X has a supremum in X. Moreover, X is of the countable type if every subset E of X that has a supremum in X, contains a countable subset $E^* \subseteq E$ so that: $\sup E = \sup E^*$.

Let X be a partially ordered semigroup and H a ring of subsets of T. The function m: $H \to X$ is an o-measure (order measure) on H, if m is positive on H ($m(A) \geqslant 0$, for every A in H) and $m(\bigcup_{n\in N} A_n) = \sup\{\sum_{i=1}^n m(A_i)$: $n \in N\}$ whenever $(A_n)_{n\in N}$ is a disjoint sequence of elements of H with $(\bigcup_{n\in N} A_n) \in H$.

The following propositions can be easily proved.

PROPOSITION 1.1. *Let m: $H \to X$ be an o-measure on H.*

(1) $m(\varnothing) = 0$.

(2) *m is finitely additive on H and $m(A) \leqslant m(B)$, whenever A, $B \in H$ with $A \subseteq B$.*

(3) *For every sequence $(A_n)_{n\in N}$ in H with $(\bigcup_{n\in N} A_n) \in H$ and $\sup\{\sum_{i=1}^n m(A_i)$: $n \in N\} \in X$, implies: $m(\bigcup_{n\in N} A_n) \leqslant \sup\{\sum_{i=1}^n m(A_i)$: $n \in N\}$.*

Received by the editors August 22, 1977 and, in revised form, November 30, 1977.

AMS (MOS) subject classifications (1970). Primary 46G99; Secondary 28A55.

Key words and phrases. Partially ordered semigroup, monotone complete partially ordered semigroup, partially ordered semigroup of the countable type, o-measure, absolutely continuous and singular o-measure, partially ordered topological semigroup, σ-compatible topology with the partial ordering, τ_X-measure.

(4) *If X is monotone complete then for every disjoint family $(A_i)_{i\in I}$ in H with $(\bigcup_{i\in I}A_i) \in H$ implies*: $m(\bigcup_{i\in I}A_i) \geqslant \sum_{i\in I}m(A_i) := \sup\{\sum_{i\in J}m(A_i)\colon J \subseteq I$, *$J$ finite*$\}$.

Proposition 1.2. *The function $m\colon H \to X$ is an o-measure on H if and only if m is positive, finitely additive on H and $m(A_n)\uparrow m(A)$ $(m(A_n) \leqslant m(A_{n+1})$, $n \in N$ and $m(A) = \sup\{m(A_n)\colon n \in N\})$, for every increasing sequence $(A_n)_{n\in N}$ in H with $A_n\uparrow A \in H$.*

2. Absolutely continuous and singular o-measures. Let X, Y be partially ordered semigroups and let $m\colon H \to X$, $l\colon H \to Y$ be o-measures on H. l is m-absolutely continuous on H $(l \ll m)$ if $l(A) = 0$ whenever $A \in H$ with $m(A) = 0$. On the other hand l is m-singular on H, $(l \perp m)$ if for every A in H there is B in H: $B \subseteq A$, $m(B) = 0$ and $l(A - B) = 0$. So $m \perp l$ if and only if $l \perp m$.

The following proposition can be easily verified.

Proposition 2.1. *Let $m\colon H \longrightarrow X$, $l\colon H \longrightarrow Y$ and $k\colon H \longrightarrow Y$ be o-measures on H.*

(1) *If $l \perp m$ and $l \ll m$ then $l = 0$.*
(2) *If $l \perp m$ and $k \ll m$ then $l \perp k$.*
(3) *$l \perp l$ if and only if $l = 0$.*
(4) *If $m \perp l$ and $m \perp k$ then $m \perp (l + k)$.*
(5) *If $l \perp m$ and $k \perp m$ then $(l + k) \perp m$.*
(6) *If $X = Y$, and $l \leqslant m + k$, $l \perp m$ then $l \leqslant k$.*

On the other hand the following lemma will be useful in the sequence.

Lemma 2.2. *Let $m_i\colon H \to X$, $i \in I$, be an increasing directed family of o-measures on H. Suppose, that X is a monotone complete partially ordered semigroup and for every $A \in H$ there is x in X such that*: $m_i(A) \leqslant x$, *whenever $i \in I$. Then the function $m\colon H \to X$, $m(A) = \sup\{m_i(A)\colon i \in I\}$ is an o-measure on H.*

Proof. Let A, $B \in H$ with $A \cap B = \varnothing$, so $m(A \cup B) = \sup\{m_i(A \cup B)\colon i \in I\} = \sup\{m_i(A) + m_i(B)\colon i \in I\} \leqslant \sup\{m_i(A)\colon i \in I\} + \sup\{m_i(B)\colon i \in I\} = m(A) + m(B)$. Furthermore let i, j be any pair of indices. Then there exist $h \in I$ such that, $h > i$ and $h > j$, hence $m_i(A) + m_j(B) \leqslant m_h(A) + m_h(B) = m_h(A \cup B) \leqslant m(A \cup B)$, which implies $m(A) + m(B) = m(A \cup B)$, namely m is finitely additive on H. Evidently $m(A) \leqslant m(B)$ whenever A, $B \in H$ with $A \subseteq B$.

Finally let $(A_n)_{n\in N}$ be a sequence in H with $A_n\uparrow A \in H$. Then $m_i(A_n)\uparrow m_i(A)$, for every i in I. Thus:

$$\sup\{m(A_n)\colon n \in N\} = \sup\{\sup\{m_i(A_n)\colon i \in I\}\colon n \in N\}, \tag{1}$$

$$m(A) = \sup\{\sup\{m_i(A_n)\colon n \in N\}\colon i \in I\}. \tag{2}$$

But $\{m_i(A_n)\colon i \in I,\ n \in N\} = \bigcup_{i\in I}\{m_i(A_n)\colon n \in N\} = \bigcup_{n\in N}\{m_i(A_n)\colon i \in I\}$, hence

$$\begin{aligned}\sup\{\sup\{m_i(A_n): i \in I\}: n \in N\} &= \sup\{\sup\{m_i(A_n): n \in N\}: i \in I\}\\ &= \sup\{m_i(A_n): i \in I, n \in N\} \qquad (3)\end{aligned}$$

(cf. [**11**, p. 12, Theorem I.6.1]). Therefore by (1), (2) and (3) it follows that $m(A_n)\uparrow m(A)$ and the assertion follows from Proposition 1.2.

Hereafter by S it is denoted a σ-ring of subsets of T.

PROPOSITION 2.3. *Let m_i: $S \to X$, $i \in I$ be an increasing directed family of o-measures on S and l: $S \to Y$ be another o-measure on S. Suppose that X is of the countable type partially ordered semigroup*, $\sup\{m_i(A): i \in I\} = m(A) \in X$, *whenever $A \in S$ and $m_i \perp l$ for every $i \in I$. Then m: $S \to X$ is an o-measure on S with $m \perp l$.*

PROOF. By Lemma 2.2 it follows that m is an o-measure on S. Now let $A \in S$. Then there is a countable subset $\{i(n): n \in N\}$ of I, such that: $m(A) = \sup\{m_{i(n)}(A): n \in N\}$. On the other hand, there is a sequence $(B_n)_{n \in N}$ in S with $B_n \subseteq A$, $m_{i(n)}(A) = m_{i(n)}(B_n)$ and $l(B_n) = 0$, for every $n \in N$. We put $B = \bigcup_{n \in N} B_n$ hence $B \subseteq A$, $m_{i(n)}(A) = m_{i(n)}(B)$ and $l(B) = 0$, $n \in N$. Consequently

$$m(A) = \sup\{m_{i(n)}(A): n \in N\} = \sup\{m_{i(n)}(B): n \in N\} \leqslant m(B) \leqslant m(A),$$

so $m(A - B) = 0$ and $l(B) = 0$.

COROLLARY 2.4. *Let m_n: $S \to X$, $n \in N$, be an increasing sequence of o-measures on S and let l: $S \to Y$ be another o-measure on S. Suppose that* $\sup\{m_n(A): n \in N\} = m(A) \in X$, *whenever $A \in S$ and $m_n \perp l$, for every $n \in N$. Then m: $S \to X$ is an o-measure on S and $m \perp l$.*

3. The Lebesgue Decomposition Theorem. First we give the following:

LEMMA 3.1. *Let m: $S \to X$ be an o-measure on the σ-ring S and let Λ be a nonempty subfamily of S closed to countable unions. Suppose that X is a monotone complete of the countable type partially ordered semigroup. Then the function m_1: $S \to X$, $m_1(A) = \sup\{m(A \cap M): M \in \Lambda\}$, is an o-measure on S and for every A in S, there exists $M \in \Lambda$ such that $m_1(A) = m(A \cap M)$.*

PROOF. Let $A \in S$. From the hypothesis it is easily verified that there exists an increasing sequence $(M_n)_{n \in N}$ in Λ with $M_n \uparrow M \in \Lambda$ and $m(A \cap M) = \sup\{m(A \cap M_n): n \in N\} = m_1(A)$.

Next let $(m_M)_{M \in \Lambda}$ be the increasing directed family of o-measures on S, such that $m_M(A) = m(A \cap M)$ whenever $M \in \Lambda$ and $A \in S$. By Lemma 2.2 and from $m_1(A) = \sup\{m_M(A): M \in \Lambda\}$, $A \in S$ it follows that m_1 is an o-measure on S.

THEOREM 3.2 (LEBESGUE DECOMPOSITION). *Let the o-measures be m: $S \to X$, l: $S \to Y$ on the σ-ring S. Suppose, Y is a monotone complete of the countable type partially ordered semigroup. Then there exist unique o-measures l_i: $S \to Y$, $i = 1, 2$, such that*:

$$l = l_1 + l_2, \quad l_1 \ll m, \quad l_2 \perp m, \quad l_1 \perp l_2.$$

PROOF. We consider the functions: l_1: $S \to Y$, l_2: $S \to Y$, $l_1(A) = \sup\{l(A \cap M)$: $M \in \Lambda\}$, $l_2(A) = \sup\{l(A \cap Q)$: $Q \in \Theta\}$ where $\Theta = \{Q \in S$: $m(Q) = 0\}$ and $\Lambda = \{M \in S$: $l_2(M) = 0\}$. Clearly from Lemma 3.1 the functions l_i: $S \to Y$, $i = 1, 2$, are o-measures on S and there exist $M \in \Lambda$, $Q \in \Theta$ such that:

$$l_1(A) = l(A \cap M) = l_1(A \cap M), \tag{4}$$

$$l_2(A) = l(A \cap Q) = l_2(A \cap Q). \tag{5}$$

If $m(A) = 0$ then $(A \cap M) \in \Theta$, hence $l_1(A) = l(A \cap M) = l_2(A \cap M) = 0$, namely $l_1 \ll m$.

On the other hand $(A - Q) \in \Lambda$ (because $(A - Q) \notin \Lambda$ implies $l_2(A - Q) > 0$, so by (5) $l_2(A) > l_2(A)$, that is a contradiction), therefore $l(A - Q) = l_1(A - Q) = l_1(A)$. Thus $l(A) = l(A - Q) + l(A \cap Q) = l_1(A) + l_2(A)$. Now from (4) and (5) one obviously has $l_1 \perp l_2$ and $l_2 \perp m$. To show uniqueness let $l = l_1 + l_2 = l_3 + l_4$ be two such decompositions. Evidently $l_4 \perp l_1$ and $l_2 \perp l_3$. So from $l_2 \leqslant l_3 + l_4$ and $l_4 \leqslant l_1 + l_2$ imply $l_2 \leqslant l_4$ and $l_4 \leqslant l_2$, hence $l_2 = l_4$. Furthermore from $l_1 \perp l_2$, $l_3 \perp l_2$, $l_1 \leqslant l_2 + l_3$ and $l_3 \leqslant l_1 + l_2$ we also have $l_1 = l_3$.

4. Partially ordered topological semigroup-valued measures. Throughout this paragraph we suppose that X is a partially ordered topological semigroup, that is a partially ordered semigroup, equipped with a Hausdorff topology τ_X such that the sets: $E_x := \{y \in X$: $y \geqslant x\}$, $F_x := \{y \in X$: $y \leqslant x\}$ are τ_X-closed, whenever $x \in X$. In this place we give the well-known lemma.

LEMMA 4.1. *Let $(x_i)_{i \in I}$ be an increasing directed family in the partially ordered topological semigroup X with τ_X-$\lim x_i = x$ (convergence in the topology τ_X of X). Then $x = \sup\{x_i$: $i \in I\}$.*

PROOF. We set $E_i = \{y \in X$: $y \geqslant x_i\}$ for every $i \in I$, hence $x \in \overline{E}_i = E_i$ (by $\overline{E}_i$ we denote the closure of E_i in X), namely $x \geqslant x_i$ for every $i \in I$. Moreover let z be an element of X so that:

$$x_i \leqslant z, \quad \text{for any } i \in I.$$

Thus by the fact that the set $F = \{y \in X$: $y \leqslant z\}$ is τ_X-closed and hypothesis, one similarly has, $x \in \overline{F} = F$, which proves the assertion. Next the topology τ_X is called σ-compatible with the partial ordering if every majorised increasing sequence in X converges relative to the topology τ_X.

Now the function m: $H \to X$ is a τ_X-measure on the ring H, if m is positive on H and m is σ-additive on H with respect to topological convergence in X. The definitions and results of absolute continuity and singularity are similar as above.

In particular we obtain.

THEOREM 4.2. *Let the τ_X-measure m: $S \to X$ and the τ_Y-measure l: $S \to Y$ on the σ-ring S. Suppose that Y is a monotone complete of the countable type* *partially ordered topological semigroup and the topology* τ_Y *is* σ*-compatible with the partial ordering. Then there exist unique* τ_Y*-measures* l_i: $S \to Y$, $i = 1, 2$, *such that*:

$$l = l_1 + l_2, \quad l_1 \ll m, \quad l_2 \perp m, \quad l_1 \perp l_2.$$

The proof of the Theorem 4.2 follows from Lemma 4.1 and Theorem 3.2.

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Department of Mathematical Analysis, Athens University, Panepistemiopolis, Athens 621, Greece