PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 2, September 1978

LOCALLY CLOSEDNESS OF UNBOUNDED DERIVATIONS IN C^*-ALGEBRAS

SCHÔICHI ÔTA

ABSTRACT. We study a local property of unbounded derivations in C^*-algebras, introducing *locally closedness* for derivations. We show that a derivation is locally closed and the positive portion of the domain is closed under the square root operation if and only if for each hermitian element a in the domain the C^*-subalgebra generated by a and the identity is contained in the domain.

1. Preliminaries. In a recent few years the theory of unbounded derivations in C^*-algebras has been investigated by several authors ([2], [3], [4], [9], [10]) and others. In this paper we shall study a derivation with a local property (locally closedness) which is weaker than closedness.

Let $\mathfrak{A}$ be a C^*-algebra with the identity on a Hilbert space $\mathcal{H}$. A linear mapping δ from the domain $\mathfrak{D}(\delta)$, which is a dense $*$-subalgebra of $\mathfrak{A}$, into $\mathfrak{A}$ is said to be a *$*$-derivation* if it satisfies two conditions

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \text{and} \quad \delta(a^*) = \delta(a)^*$$

for $a \in \mathfrak{D}(\delta)$ and $b \in \mathfrak{D}(\delta)$.

It is known [9] that δ is closable if the positive portion of $\mathfrak{D}(\delta)$ is closed under the square root operation. On the other hand we have shown in [8] that, if δ is closed and the positive portion of $\mathfrak{D}(\delta)$ is closed under the square root operation, then δ is bounded (i.e., $\mathfrak{D}(\delta) = \mathfrak{A}$). Introducing the property of locally closedness, we shall first characterize the property of the domain for which we may exchange a role of closedness for that of locally closedness in this situation (Theorem 4). Next we shall consider the relation between closability and locally closedness.

In this paper we shall use the following special representation of the domain $\mathfrak{D}(\delta)$ on some Hilbert space. Let δ be a $*$-derivation in a C^*-algebra $\mathfrak{A}$ on $\mathcal{H}$. We shall define the mapping π_δ of $\mathfrak{D}(\delta)$ into the total operator algebra $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ on two fold copies $\mathcal{H} \oplus \mathcal{H}$ of $\mathcal{H}$ as follows:

$$a \in \mathfrak{D}(\delta) \to \pi_\delta(a) = \begin{pmatrix} a & 0 \\ \delta(a) & a \end{pmatrix}.$$

Let J be an hermitian unitary operator on $\mathcal{H} \oplus \mathcal{H}$ defined by $J(\xi \oplus \eta) = \eta \oplus \xi$ for $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}$. Then if δ is closed, by [7, Proposition 2.1], the

Received by the editors September 21, 1977.

AMS (MOS) subject classifications (1970). Primary 46L05.

Key words and phrases. C^*-algebras, unbounded derivation, closability, locally closedness.

algebra $\pi_\delta(\mathfrak{D}(\delta)) = \{\pi_\delta(a): a \in \mathfrak{D}(\delta)\}$ is a *semisimple* involutive Banach algebra with the operator norm on $\mathcal{H} \oplus \mathcal{H}$ and the *hermitian* involution defined by

$$\pi_\delta(a) \to \pi_\delta(a)^J = J\pi_\delta(a)^* J.$$

One may also consider the algebra $\pi_\delta(\mathfrak{D}(\delta))$ as an operator algebra on an indefinite inner product space. We leave to the references **[6]**, **[7]** for details.

We denote the spectrum of an element a in a Banach algebra $\mathfrak{C}$ by $\mathrm{Sp}(a$: $\mathfrak{C})$. We simply write $\mathrm{Sp}(a)$ in place of $\mathrm{Sp}(a$: $\mathfrak{A})$ whenever $\mathfrak{A}$ is a C^*-algebra.

The author is deeply indebted to Professor J. Tomiyama for discussions.

2. Definition and root operation.

Definition 1. Let $\mathfrak{A}$ be a C^*-algebra with the identity on a Hilbert space $\mathcal{H}$ and δ be a $*$-derivation in $\mathfrak{A}$. We say that δ is *locally closed* if the restriction of δ to the $*$-subalgebra $C(a) \cap \mathfrak{D}(\delta)$ is closed for each hermitian $a \in \mathfrak{D}(\delta)$, which means that if $\lim a_n = b$ and $\lim \delta(a_n) = c$ for a sequence $\{a_n\}$ in $C(a) \cap \mathfrak{D}(\delta)$ then this implies that $b \in C(a) \cap \mathfrak{D}(\delta)$ and $\delta(b) = c$. Here, we mean by $C(a)$ the C^*-subalgebra of $\mathfrak{A}$ generated by a and the identity 1.

We remark that a locally closed $*$-derivation is not always closed. In fact, a normal $*$-derivation in a UHF algebra (see for the definition, **[9]**) is locally closed but not closed.

We begin with an improvement of the result known for a closed $*$-derivation **[2]**, **[3]**, **[4]**.

Proposition 2. *Let $\mathfrak{A}$ be a C^*-algebra with the identity on $\mathcal{H}$ and δ be a locally closed $*$-derivation in $\mathfrak{A}$. Then we have*

1. *The identity is contained in the domain $\mathfrak{D}(\delta)$.*

2. *For an hermitian $a \in \mathfrak{D}(\delta)$ and a complex valued function $f(x)$ which is analytic in a simply connected domain Γ containing* $\mathrm{Sp}(a)$

$$f(a) = \frac{1}{2\pi i}\oint_\partial f(\lambda)(\lambda - a)^{-1}\, d\lambda$$

is contained in $\mathfrak{D}(\delta)$ where ∂ is a closed rectifiable Jordan curve in Γ with $\mathrm{Sp}(a)$ *contained in its interior.*

Proof. The proofs are similar to Chi's argument **[4**, Lemma 3.3 and Theorem 3.2], and so we shall only give the proof of 2. There exists a sequence $\{P_n(x)\}$ of polynomials such that $\{P_n(x)\}$ converges to $f(x)$ uniformly as $n \to \infty$ in a domain contained in Γ and containing $\mathrm{Sp}(a)$. By a simple calculation we have that $\mathrm{Sp}(a) = \mathrm{Sp}(\pi_\delta(a)$: $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H}))$ (cf. **[4**, Lemma 3.2]) and so $\{P_n(a)\}$ converges to $f(a)$ and also $\{\pi_\delta(P_n(a))\}$ $(= \{P_n(\pi_\delta(a))\})$ converges in norm to $f(\pi_\delta(a))$ as $n \to \infty$. From the inequality

$$\|\pi_\delta(P_n(a))\| \geqslant \max\{\|P_n(a)\|, \|\delta(P_n(a))\|\},$$

both sequences $\{P_n(a)\}$ and $\{\delta(P_n(a))\}$ become Cauchy sequences. Since $P_n(a) \in C(a) \cap \mathfrak{D}(\delta)$ and δ is locally closed we see that $f(a) \in C(a) \cap \mathfrak{D}(\delta)$ and $\pi_\delta(f(a)) = f(\pi_\delta(a))$.

REMARK 3. If δ is closed then $\mathrm{Sp}(a) = \mathrm{Sp}(\pi_\delta(a)\colon \pi_\delta(\mathfrak{D}(\delta)))$ for each hermitian $a \in \mathfrak{D}(\delta)$. In fact, since π_δ is an isomorphism of $\mathfrak{D}(\delta)$ onto $\pi_\delta(\mathfrak{D}(\delta))$ we have $\mathrm{Sp}(a) \subset \mathrm{Sp}(\pi_\delta(a)\colon \pi_\delta(\mathfrak{D}(\delta)))$. Conversely suppose that $\lambda \in \mathbf{C} \setminus \mathrm{Sp}(a)$, then we have $(\lambda - a)^{-1} \in \mathfrak{D}(\delta)$ by Proposition 2 and hence $\lambda - \pi_\delta(a)$ has an inverse element in $\pi_\delta(\mathfrak{D}(\delta))$ with $(\lambda - \pi_\delta(a))^{-1} = \pi_\delta((\lambda - a)^{-1})$.

THEOREM 4. *Let $\mathfrak{A}$ be a C^*-algebra with the identity acting on a Hilbert space $\mathcal{H}$ and δ be a $*$-derivation in $\mathfrak{A}$. Then the following conditions are equivalent.*

(A) *For each hermitian $a \in \mathfrak{D}(\delta)$, $C(a)$ is contained in $\mathfrak{D}(\delta)$.*

(B) *δ is locally closed and the positive portion of $\mathfrak{D}(\delta)$ is closed under the square root operation.*

PROOF. Suppose that the condition (A) is satisfied. Then it is clear that the positive portion of $\mathfrak{D}(\delta)$ is closed under the square root operation and so δ is closable ([**9**]). It follows that the condition (A) implies (B). Conversely suppose that the condition (B) is satisfied. Take an hermitian a in $\mathfrak{D}(\delta)$. Suppose $\{\pi_\delta(a_n)\}$ is a Cauchy sequence in $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$. Then both sequences $\{a_n\}$ and $\{\delta(a_n)\}$ are Cauchy sequences. Since δ is locally closed there is an element $b \in C(a) \cap \mathfrak{D}(\delta)$ such that $\delta(b) = \lim \delta(a_n)$, hence $\{\pi_\delta(a_n)\}$ converges to $\pi_\delta(b)$. Therefore, one sees that $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$ is a commutative involutive Banach algebra which is, as easily seen, semisimple. Now the domain $\mathfrak{D}(\delta)$ is closed under the square root operation of positive elements and $\mathrm{Sp}(k) = \mathrm{Sp}(\pi_\delta(k)\colon \pi_\delta(C(a) \cap \mathfrak{D}(\delta)))$ for each hermitian $k \in \mathfrak{D}(\delta) \cap C(a)$ by Remark 3. The algebra $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$ has an hermitian involtion and is closed under the square root operation of hermitian elements with positive spectrum. It follows from *Katznelson's Theorem* [**6**] that $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$ is C^*-equivalent, that is, there is an involution-preserving isomorphism of $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$ onto a C^*-algebra $\mathfrak{B}$. Since π_δ is also an involution-preserving isomorphism, i.e., $\pi_\delta(a^*) = \pi_\delta(a)^J$, there exists a $*$-isomorphism of the C^*-algebra $\mathfrak{B}$ onto a $*$-subalgebra $C(a) \cap \mathfrak{D}(\delta)$ of $\mathfrak{A}$. Therefore, the image $C(a) \cap \mathfrak{D}(\delta)$ is a C^*-subalgpbra of $\mathfrak{A}$. As $C(a) \cap \mathfrak{D}(\delta)$ is dense in $C(a)$, the algebra $C(a)$ is contained in $\mathfrak{D}(\delta)$. The proof is complete.

REMARK 5. Suppose that δ satisfies the condition in Theorem 4. Then the domain $\mathfrak{D}(\delta)$ has the property that *for any closed ideal I of $\mathfrak{A}$ the algebra $I \cap \mathfrak{D}(\delta)$ is dense in I.* The proof is the same as in [**1**, Lemma 3].

3. Closability. Let δ be a locally closed $*$-derivation in a C^*-algebra $\mathfrak{A}$ with the identity on a Hilbert space $\mathcal{H}$. For any given projection p of $\mathfrak{A}$ and $\varepsilon > 0$, following [**2**], [**10**] there is an hermitian $a \in \mathfrak{D}(\delta)$ such that $\|p - a\| < \varepsilon' = \min\{1/14, \varepsilon/8\}$ and so $\mathrm{Sp}(a) \subset [-6\varepsilon', 6\varepsilon'] \cup [1 - 6\varepsilon', 1 + 6\varepsilon']$. Let γ be a circle in $\mathbf{C}$ with the center 1 and the radius $1/2$. Put

$$q = \frac{1}{2\pi i} \oint_\gamma (\lambda - a)^{-1}\, d\lambda.$$

We can see that q is a projection with $\|p - q\| < 7\varepsilon' < \varepsilon$ ([**2**, the proof of Theorem 3] or [**10**, the proof of Lemma 1]). We shall show that q belongs to $\mathfrak{D}(\delta)$. Since $\pi_\delta(C(a) \cap \mathfrak{D}(\delta))$ is an involutive Banach algebra and $\mathrm{Sp}(a) = \mathrm{Sp}(\pi_\delta(a)\colon \pi_\delta(C(a) \cap \mathfrak{D}(\delta)))$ (the proof of Theorem 4), we can define

$$Q = \frac{1}{2\pi i}\oint_\gamma (\lambda - \pi_\delta(a))^{-1}\, d\lambda \in \pi_\delta(C(a) \cap \mathfrak{D}(\delta)).$$

Thus the Riemann sums for the integral $(1/2\pi i)\oint(\lambda - a)^{-1}\, d\lambda$, $\sum_{k=1}^{k(n)} \alpha_k^{(n)}(\lambda_k^{(n)} - a)^{-1}$ converges to q as well as the sums

$$\sum_{k=1}^{k(n)} \alpha_k^{(n)}\left(\lambda_k^{(n)} - \pi_\delta(a)\right)^{-1} = \pi_\delta\left(\sum_{k=1}^{k(n)} \alpha_k^{(n)}\left(\lambda_k^{(n)} - a\right)^{-1}\right)$$

converging to Q. It follows that the sequence $\{\delta(a_n)\}$ is a Cauchy sequence where

$$a_n = \sum_{k=1}^{k(n)} \alpha_k^{(n)}\left(\lambda_k^{(n)} - a\right)^{-1}.$$

By Proposition 2, $a_n \in C(a) \cap \mathfrak{D}(\delta)$ $(n = 1, 2, \ldots)$. Hence, q belongs to $D(\delta)$ from the locally closedness of δ.

In the above situation suppose that $\mathfrak{A}$ contains the C^*-algebra of all compact operators on $\mathcal{H}$. Then it is easily seen that $\mathfrak{D}(\delta)$ contains a finite rank operator by the above argument and hence δ is closable by [**2**, Theorem 8]. Thus we have obtained

PROPOSITION 6. *Let $\mathfrak{A}$ be a C^*-algebra with the identity on a Hilbert space $\mathcal{H}$, which contains the C^*-algebra of all compact operators on $\mathcal{H}$. Then a locally closed $*$-derivation in $\mathfrak{A}$ is closable.*

REMARK 7. The author does not know whether or not locally closedness implies closability. A normal $*$-derivation in a UHF algebra is not only locally closed but also closable.

REFERENCES

1. H. Benche and J. Cuntz, *Locally completeness of operator algebras*, Proc. Amer. Math. Soc. **62** (1977), 95–100.

2. O. Bratteli and D. W. Robinson, *Unbounded derivations of C*-algebras*, Comm. Math. Phys. **42** (1975), 253–268. MR **51** #13698.

3. ______, *Unbounded derivations of C*-algebras*. II, Comm. Math. Phys. **46** (1976), 11–30. MR **52** #11608.

4. D. P. Chi, *Derivations in C*-algebras* (preprint).

5. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

6. Y. Katznelson, *Sur les algèbres dont les éléments non-négatifs admettent des racines carrées*, Ann. Sci. École Norm. Sup. **77** (1960), 167–174. MR **22** #12403.

7. S. Ôta, *On a representation of a C*-algebra in a Lorentz algebra*, Acta Sci. Math (Szeged) **39** (1977), 129–133.

8. ______, *Certain operator algebras induced by *-derivations in C*-algebras on an indefinite inner product space*, J. Functional Analysis (to appear).

9. R. T. Powers and S. Sakai, *Unbounded derivations in operator algebras*, J. Functional Analysis **19** (1975), 81–95. MR **52** #3974.

10. S. Sakai, *On one-parameter subgroups of *-automorphisms on operator algebras and the corresponding unbounded derivations*, Amer. J. Math. **98** (1976), 427–440.

DEPARTMENT OF MATHEMATICS, YAMAGATA UNIVERSITY, KOSHIRAKAWA, YAMAGATA 990, JAPAN

Current address: Department of Mathematics, Kyushu University, Fukuoka, 812 JAPAN