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BALAYAGE BY FOURIER TRANSFORMS WITH SPARSE FREQUENCIES IN COMPACT ABELIAN TORSION GROUPS

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ABSTRACT. Let Λ be a discrete subset of a LCA group and E a compact subset of the dual group Γ. Balayage is said to be possible for (Λ, E) if the Fourier transform of each measure on G is equal on E to the Fourier transform of some measure supported by Λ.

For a class of infinite compact metrizable Γ, including all such torsion groups, we show how to construct $E \subset \Gamma$ such that there are arbitrarily sparse sets Λ with balayage possible for (Λ, E). E is, moreover, large enough that the set of products $E \cdot E \cdot E = \Gamma$.

Let G be a locally compact abelian group with dual group Γ, both written multiplicatively. For $\mu \in M(G)$, we denote by $\hat{\mu}$ the inverse Fourier transform $\hat{\mu}(\gamma) = \int_G \langle \gamma, x \rangle \, d\mu(x)$.

We recall a notion introduced by Beurling [**1**].

DEFINITION 1. If $\Lambda \subset G$ is discrete and $E \subset \Gamma$ is compact, then balayage is said to be possible for (Λ, E) if, for every $\mu \in M(G)$ there is some $\nu \in M(\Lambda)$ with $\hat{\nu}(\gamma) = \hat{\mu}(\gamma)$ for all $\gamma \in E$.

In [**3**, p. 151] and [**4**, p. 160], Kahane shows the existence of fairly large sets E (of Cantor type) in the circle group T with the following property. Given any sequence $\{\phi_n\}_1^\infty$ of real-valued functions (ϕ_n having $n-1$ arguments) there is a set $\Lambda = \{\lambda_n\}_1^\infty \subset Z$ with balayage possible for (Λ, E) and with $\lambda_n \geqslant \phi_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ for all n. (Kahane does not use the terminology of balayage.) In [**5**], similar examples of such sets E were constructed, in the real line, using a different technique. Such sets were called BAS sets (for "balayage with arbitrarily sparse frequencies"). The purpose of this paper is to define the notion of BAS sets for general LCA groups, and to give, for metrizable compact torsion groups, fairly "large" examples of such sets. The technique used here involves a necessary modification of that of [**5**].

DEFINITION 2. $E \subset \Gamma$ is called a BAS set if, given any sequence of functions, $\{F_n\}_1^\infty$, where F_n takes values in the collection of compact subsets of G, there is a set $\Lambda = \{\lambda_n\}_1^\infty$ with balayage possible for (Λ, E) and with

$$\lambda_n \notin F_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \tag{1}$$

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for all n. (Here, F_1 is just a compact subset of G.)

Let $B(E)$ be the Banach algebra of restrictions to E of Fourier transforms of measures in $M(G)$ normed by

$$\|\phi\|\widehat{B(E)} = \inf\{\|\mu\|: \mu \in M(G) \text{ and } \hat{\mu}|E = \phi\}.$$

It is straightforward to modify [5, Theorem 6.1], which was stated only for the real case, to prove

THEOREM 1. *If there is some $\delta < 1$ and a sequence $\{t_n\}_1^\infty \subset G$ with $\lim_{n\to\infty} t_n = \infty$ and with*

$$\|1 - \langle \cdot, t_n\rangle\|_{B(E)} < \delta$$

for all n, then E is a BAS set. (Here, $\langle\cdot, x\rangle$ is the function $\gamma \mapsto \langle\gamma, x\rangle$.)

Theorem 1 suffices for constructing examples of fairly large BAS sets in many metrizable groups (for example, the groups of p-adic integers). Suppose, however, that every element of Γ has order at most 6. Then, for $x \in G$ and $\gamma \in \Gamma$, if $\langle\gamma, x\rangle \neq 1$, then $\langle\gamma, x\rangle$ is a nontrivial nth root of 1 for some $n \leqslant 6$ and, thus,

$$|1 - \langle\gamma, x\rangle| \geqslant 1.$$

It is, therefore, impossible for the hypotheses of Theorem 1 to apply to E unless $\langle\gamma, t_n\rangle = 1$ for all $\gamma \in E$. In such a case, E must be quite thin. (For example, no finite product $E \cdot E \cdot \ldots \cdot E$ can have interior.) Via a somewhat different route, however, we can prove

THEOREM 2. *If Γ is an infinite, metrizable, compact, abelian, torsion group, then Γ contains a BAS set E with $E \cdot E \cdot E = \Gamma$.*

PROOF. By [2, 25.9], we may write Γ as a product

$$\Gamma = \bigtimes_{n=1}^{\infty} \Gamma_n$$

where each Γ_n is a nontrivial, finite group (and Γ has the product topology). Then G is a discrete direct sum

$$G = \bigoplus_{n=1}^{\infty} G_n$$

where each G_n is the dual group of Γ_n (and so is isomorphic to Γ_n). Here typical elements x and γ of G and Γ respectively are sequences

$$x = \langle x(1), x(2), x(3), \ldots\rangle \quad \text{and} \quad \gamma = \langle\gamma(1), \gamma(2), \gamma(3), \ldots\rangle$$

where each $x(n) \in G_n$ and $\gamma(n) \in \Gamma_n$ and all but finitely many of the $x(n)$ equal 1. The duality between G and Γ is given by

$$\langle\gamma, x\rangle = \prod_1^\infty \langle\gamma(n), x(n)\rangle$$

where all but finitely many terms in the product are 1.

For $r = 1, 2, 3$, let

$$I_r = \{n \geqslant 1: n \equiv r \pmod 3\}$$

and let

$$E_r = \{\gamma \in \Gamma: \gamma(n) = 1 \text{ if } n \notin I_r\}.$$

Finally, we let $E = E_1 \cup E_2 \cup E_3$. We note that it is clear that $E \cdot E \cdot E = \Gamma$.

To verify that E is a BAS set requires some auxiliary constructions which show the possibility of approximating the constant function 1 on E by a linear combination of characters (instead of just by characters as in Theorem 1).

s_m will denote an element of G such that
$s_m(n) = 1$ if $n \neq m$ and
$s_m(m) \neq 1$.
(The exact choice of $s_m(m)$ is irrelevant. What we need is that the various s_m be distinct.)

$\delta(x)$ denotes the unit point mass at x.

We shall use the letters i, j and k (subscripted or not) to denote elements of I_1, I_2 and I_3 respectively. With this convention in mind, we define

$$\mu = \mu(i, j, k) = \tfrac{1}{3}\left(\delta(s_i) + \delta(s_j) + \delta(s_k)\right)$$

and

$$\nu = \nu(i, j, k) = \tfrac{1}{3}\left(\delta(0) - \delta(s_i s_j s_k)\right).$$

We note that if $\gamma \in E$ then at most one of $\langle \gamma, s_i \rangle$, $\langle \gamma, s_j \rangle$ and $\langle \gamma, s_k \rangle$ can fail to equal 1. Thus, for $\gamma \in E$,

$$\begin{aligned} 1 - \hat{\mu}(\gamma) &= 1 - \tfrac{1}{3}\left(\langle \gamma, s_i \rangle + \langle \gamma, s_j \rangle + \langle \gamma, s_k \rangle\right) \\ &= \tfrac{1}{3} - \tfrac{1}{3}\langle \gamma, s_i \rangle \langle \gamma, s_j \rangle \langle \gamma, s_k \rangle = \hat{\nu}(\gamma). \end{aligned}$$

Since $\|\nu(i, j, k)\| = \frac{2}{3}$ we conclude that

$$\left\| 1 - \mu(i, j, k)^\wedge \right\|_{B(E)} \leqslant \tfrac{2}{3}. \tag{2}$$

Enumerate $G = \{g_n\}_1^\infty$.

Suppose functions $\{F_n\}_1^\infty$ as in Definition 2 are given. We construct Λ inductively, adding three elements to Λ for each element of G. Note that Sx^{-1} denotes $\{sx^{-1}: s \in S\}$. The fact that each F_n takes values in the collection of finite subsets of G should be borne in mind.

Pick $i_1 \in I_1$ such that $s_{i_1} \notin F_1 g_1^{-1}$ and set $\lambda_1 = s_{i_1} g_1$.
Pick $j_1 \in I_2$ such that $s_{j_1} \notin F_2(\lambda_1) g_1^{-1}$ and set $\lambda_2 = s_{j_1} g_1$.
Pick $k_1 \in I_3$ such that $s_{k_1} \notin F_3(\lambda_1, \lambda_2) g_1^{-1}$ and set $\lambda_3 = s_{k_1} g_1$.
Pick $i_2 \in I_2$ such that $s_{i_2} \notin F_4(\lambda_1, \lambda_2, \lambda_3) g_2^{-1}$ and set $\lambda_4 = s_{i_2} g_2$
and continue in this manner.

Let $\Lambda = \{\lambda_n\}_1^\infty$. Clearly, Λ satisfies (1). We must show that balayage is possible for (Λ, E). To this end, let $B_\Lambda(E) = \{\hat{\beta}|E: \beta \in M(\Lambda)\} \subset B(E)$. $B_\Lambda(E)$ becomes a Banach space under the norm

$$\|\psi\|_{B_\Lambda(E)} = \inf\{\|\beta\|: \beta \in M(\Lambda) \text{ and } \hat{\beta}|E = \psi\}.$$

It clearly suffices to show that $B_\Lambda(E) = B(E)$.

Suppose $\phi \in B(E)$. Then there is some $\alpha = \sum a_n\delta(g_n) \in M(G)$ with $\hat{\alpha}|E = \phi$ and with $\sum|a_n| = \|\alpha\| \leqslant \frac{9}{8}\|\phi\|_{B(E)}$.

Let

$$\beta = \sum a_n\delta(g_n) * \mu(i_n, j_n, k_n).$$

Then, $\beta \in M(\Lambda)$ because $\delta(g_n) * \mu(i_n, j_n, k_n)$ is supported by $\{s_{i_n}g_n, s_{j_n}g_n, s_{k_n}g_n\} = \{\lambda_{3n-2}, \lambda_{3n-1}, \lambda_{3n}\}$. Set $\psi = \hat{\beta}|E$. Then $\psi \in B_\Lambda(E)$. Noting that $\|\mu(i_n, j_n, k_n)\| = 1$, we see that

$$\|\psi\|_{B_\Lambda(E)} \leqslant \sum |a_n| \leqslant \tfrac{9}{8}\|\phi\|_{B(E)}.$$

We have also

$$\|\phi - \psi\|_{B(E)} = \left\|\sum a_n\langle\cdot, g_n\rangle(1 - \mu(i_n, j_n, k_n))^\wedge\right\|_{B(E)}$$
$$\leqslant \tfrac{2}{3}\sum |a_n| \leqslant \tfrac{3}{4}\|\phi\|_{B(E)}$$

where we have used (2) and the fact that $B(E)$ is a Banach algebra. A standard iteration now shows that $\phi \in B_\Lambda(E)$ and, indeed,

$$\|\phi\|_{B_\Lambda(E)} \leqslant \tfrac{9}{2}\|\phi\|_{B(E)} \tag{3}$$

and the theorem follows. (The point is that $\frac{9}{2} = \frac{9}{8}\sum_0^\infty(\frac{3}{4})^n$.)

REMARKS. 1. Replacing $\frac{9}{8}$ by $1 + \varepsilon$ for $\varepsilon > 0$ shows that we can replace the factor $\frac{9}{2}$ (in (3)) by 3.

2. The same result and proof hold in the more general case where the various Γ_n are merely compact and metrizable (the latter being required to make G countable).

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