ON A. HURWITZ’ METHOD IN ISOPERIMETRIC INEQUALITIES

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Abstract. We show that if $M$ is complete simply connected with nonpositive sectional curvatures, $\Omega$ a minimal submanifold of $M$ with connected suitably oriented boundary $\Gamma$ then $\sqrt[\frac{1}{2}]{\lambda V/A} < (n-1)^{\frac{1}{2}}/\pi$ where $V$ is the volume of $\Omega$, $A$ the volume of $\Gamma$, $\lambda$ the first nonzero eigenvalue of the Laplacian of $\Gamma$, and $n (> 2)$ is the dimension of $\Omega$.

In this note we prove the following extension of A. Hurwitz’ [7] argument in his proof of the classical isoperimetric inequality.

Theorem 1. Let $M$ be an $m$-dimensional complete simply connected Riemannian manifold all of whose sectional curvatures are nonpositive. Let $\Omega$ be an $n$-dimensional, $n \geq 2$, submanifold of $M$ with suitably oriented and connected boundary $\Gamma$. If $n < m$ assume $\Omega$ is minimal in $M$, i.e., its mean curvature vector vanishes identically. Let $V$ denote the $n$-volume of $\Omega$, $A$ the $(n-1)$-volume of $\Gamma$, and $\lambda$ the first nonzero eigenvalue of the Laplacian acting on functions on $\Gamma$. Then

$$\sqrt[\frac{1}{2}]{\lambda V/A} < \sqrt{n-1} / \pi. \quad (1)$$

Clearly the inequality (1) is sharp, since any $n$-disk in $\mathbb{R}^m$ will yield equality in (1). We shall present a characterization of equality in (1) for two cases only.

Theorem 2. If $n = m$ then equality in (1) implies that $\Omega$ is isometric to a disk in $\mathbb{R}^n$ endowed with the usual flat metric.

If $n < m$ and $M$ is $\mathbb{R}^m$ endowed with the usual flat metric then equality in (1) implies that $\Omega$ is the intersection of a disk in $\mathbb{R}^m$ with an $n$-dimensional affine space.

In particular, Theorems 1 and 2 recapture T. Carleman’s theorem [2] (also cf. [3]), that for a minimal surface $\Omega$ in $\mathbb{R}^3$ with boundary $\Gamma$ consisting of a smooth Jordan curve, we have the inequality

$$L^2 - 4\pi A > 0 \quad (2)$$

where, just for the moment, $A$ is the area of $\Omega$ and $L$ the length of $\Gamma$. The
point is that since $\Gamma$ is 1-dimensional Wirtinger’s inequality implies $\lambda = 4\pi^2 / L^2$. Equality in (2) is achieved if and only if $\Omega$ is a flat disk in $R^3$.

We note that the first to free Carleman’s result and its generalizations from considerations of complex function theory was W. T. Reid [9], who proved the result in 3-space via J. H. Jellet’s formula (cf. (4) below), and C. C. Hsiung [6] who generalized Reid’s argument to surfaces in $n$-space. We wish to thank R. Osserman for helpful conversations, guesses, and references. We mention R. C. Reilly [10, esp. Corollary 1] has also obtained results using A. Hurwitz’ method as the starting point. Finally we refer the reader to the inequalities of D. Hoffman and J. Spruck [5] which, although not sharp, relate the volume and area alone.

1. The inequality. The manifolds $M$, $\Omega$, $\Gamma$ will be $C^\infty$, as will be the Riemannian metric under consideration. In general we will use differentiable for $C^\infty$.

We require some notation. Let $TM$ denote the tangent bundle of $M$, and let $\nabla$ denote covariant differentiation in $M$. For tangent vectors $\xi$, $\eta$ in the same fiber in $TM$ we denote their inner product by $\langle \xi, \eta \rangle$ and the associated norm of $\xi$, by $|\xi|$.

For an open set $U$ in $\Omega$ with coordinates chart $x: U \rightarrow R^n$ we let $\{\partial_1^x, \ldots, \partial_n^x\}$ denote the natural basis of tangent spaces to $\Omega$ at points of $U$ associated with $x$. We set

$$h_{jk} = \langle \partial_j^x, \partial_k^x \rangle, \quad \Omega = (h_{jk}), \quad \Omega^{-1} = (h^{jk}),$$

$j, k = 1, \ldots, n$. The mean curvature vector of $\Omega$ in $M$ will be denoted by $H$ (if $n = m$ then $H = 0$ by definition), $dV$ will denote the volume element of $\Omega$, and $dA$ that of $\Gamma$.

For a differentiable vector field $\xi: \Omega \rightarrow TM$ ($\Omega = \Omega \cup \Gamma$) on $\Omega$ we let $\xi^T$ denote the projection of $\xi$ onto $T\Omega$. For any open neighborhood $U$ in $\Omega$ with coordinate chart $x$ as above we define the divergence of $\xi$, $\text{div}_\Omega \xi$, by

$$\text{div}_\Omega \xi = \sum_{j,k=1}^n h^{jk} \langle \nabla \partial_j^x \xi, \partial_k^x \rangle.$$

It is standard that the above definition is independent of the coordinate chart on $\Omega$. Furthermore the infinitesimal version of the first variation of area reads as [5, p. 719]

$$\text{div}_\Omega \xi^T = \text{div}_\Omega \xi + \langle \xi, H \rangle.$$  \hspace{1cm} (3)

Thus if $v$ denotes the outward normal vector field of $\Gamma$ with respect to $\Omega$ then

$$\int \int_{\Omega} \{\text{div}_\Omega \xi + \langle \xi, H \rangle\} dV = \int_{\Gamma} \langle \xi, v \rangle dA. \hspace{1cm} (4)$$

We note that if $M$ is $R^n$ with the flat metric and $X$ is the position vector of points in $R^m$ suitably identified with a tangent vector field on $R^m$ then for $\xi = X|_{\Omega}$ we have $\text{div}_\Omega \xi = n$ on all of $\Omega$ and the resulting formula (4) is that of J. H. Jellet [8].
Next we let \( p \in M, \{e_1, \ldots, e_m\} \) be an orthonormal basis of \( M_p \) and \( y: M \to R^m \) the Riemannian normal coordinates on \( M \) determined by \((p; e_1, \ldots, e_m)\). Our assumptions concerning \( M \) imply by the Hadamard-Cartan theorem that \( y \) is indeed defined on all of \( M \) and is a diffeomorphism of \( M \) onto \( R^m \). It is standard that geodesics emanating from \( p \) map onto rays emanating from the origin of \( R^m \).

**Lemma 1.** We may choose \( p, \{e_1, \ldots, e_m\} \) so that the respective coordinate functions \( y^D: M \to R, D = 1, 2, \ldots, m \), of \( y: M \to R^m \) satisfy

\[
\int \sum_{D=1}^{m} y^D \, dA = 0. \tag{5}
\]

**Proof.** Our argument is an easy adaptation of one given by H. F. Weinberger [13, p. 635].

Parallel translate the frame \( \{e_1, \ldots, e_m\} \) along every geodesic emanating from \( p \) and thereby obtain a differentiable orthonormal frame field \( \{E_1, \ldots, E_m\} \) on \( M \). Let \( y_q: M \to R^m \) denote the Riemann normal coordinates of \( M \) determined by \( \{E_1, \ldots, E_m\} \) at \( q \), and let \( (y_q)^D, D = 1, \ldots, m \) be the coordinate functions of \( y_q \). Then

\[
Y(q) = \sum_{D=1}^{m} \left( \int y^D q \, dA \right) E_D(q)
\]

is a continuous vector field on \( M \). If we restrict \( Y \) to a geodesic disk \( B \) containing \( \Omega \) then the convexity of \( B \) implies that on the boundary of \( B \), \( Y \) points into \( B \). The Brouwer fixed point theorem then implies that \( Y \) has zero on \( B \).

So we may assume that \( p, \{e_1, \ldots, e_m\} \) actually satisfies (5). Let \( X \) be the vector field on \( M \) given by

\[
X = \sum_{D=1}^{m} y^D \partial_D y
\]

where \( \{\partial_D y, D = 1, \ldots, m\} \) is the natural basis of tangent spaces associated with the coordinate chart \( y \). Of course if \( M \) is \( R^m \) with its usual flat metric then \( X \) is naturally identified with the position vector. Next set \( \xi = X|\Omega \), i.e. the restriction of \( X \) to \( \Omega \). Then standard arguments using the Rauch comparison theorem [5, p. 721] imply

\[
n \leq \text{div}_\Omega \xi. \tag{6}
\]

Thus, if \( H = 0 \) on \( \Omega \), then (4), (5), (6) imply

\[
nV \leq \int_{\Gamma} \langle \xi, v \rangle \, dA < \int_{\Gamma} |\xi| \, dA
\]

\[
\leq A^{1/2} \left( \int |\xi|^2 \, dA \right)^{1/2} = A^{1/2} \left( \int \sum_D (y^D)^2 \, dA \right)^{1/2}
\]

\[
\leq (A/\lambda)^{1/2} \left( \int \sum_D |\text{grad}_\Gamma y^D|^2 \, dA \right)^{1/2}
\]
The expression \( \nabla_T y^D \) denotes the gradient of \((y^D|\Gamma)\) in \(\Gamma\). The last inequality is, combined with (5), Lord Rayleigh's characterization of the first nonzero eigenvalue \(\lambda\) of the Laplacian on \(\Gamma\).

It remains to verify the estimate

\[
\sum_D |\nabla_T y^D|^2 < n - 1
\]

on all of \(\Gamma\).

Let \( q \in \Gamma, u : G \to \mathbb{R}^{n-1} \) be a coordinate chart on \(\Gamma\) about \(q\), \(\{\partial^u_\alpha, \alpha = 1, \ldots, n-1\}\) the natural basis of tangent spaces to \(\Gamma\) at points of \(G\) such that \(\{\partial^u_1, \ldots, \partial^u_{n-1}\}\) is orthonormal at \(q\). For \(y : M \to \mathbb{R}^m\) let \(g_{AB} = \langle \partial^u_\alpha, \partial^u_\beta \rangle\); then the Rauch comparison theorem implies that the eigenvalues of \((g_{AB})\) are all \(> 1\). Thus at \(q\)

\[
\sum_{D=1}^m |\nabla_T y^D|^2 = \sum_{D=1}^m \sum_{\alpha=1}^{n-1} \left( \frac{\partial y^D}{\partial u^\alpha} \right)^2 = \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^m \frac{\partial y^C}{\partial u^\alpha} \delta_{CD} \frac{\partial y^D}{\partial u^\alpha} < \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^m \frac{\partial y^C}{\partial u^\alpha} g_{CD} \frac{\partial y^D}{\partial u^\alpha} = \sum_{\alpha=1}^{n-1} \langle \partial^u_\alpha, \partial^u_\alpha \rangle = n - 1,
\]

and Theorem 1 is proven.

2. The case of equality for \(m = n\). If \(m = n\) and we have equality in (1) then \(|\xi|\) is constant on \(\Gamma\) from which one concludes that \(\Gamma\) is a geodesic sphere bounding the geodesic disk \(\Omega\). Furthermore \(\text{div}_\Omega \xi = n\) on all of \(\Omega\). But this in turn implies by [1, pp. 253–257; 5, ibid.] that \(\Omega\) is isometric to a disk in \(\mathbb{R}^n\).

3. The case of equality for \(n < m\), \(M = \mathbb{R}^m\). Since \(M\) is \(\mathbb{R}^m\) we identify \(\xi\) at \(q\) with the position vector at \(q\), and note that equality in (1) implies that \(\Gamma\) is contained in some sphere \(S^{m-1}(\rho)\) of radius \(\rho\) about the origin and that, furthermore, that the outward normal, \(v\), of \(\Gamma\) with respect to \(\Omega\) is \(\rho^{-1} \xi\).

Next we note that equality in (1) implies, by the Rayleigh characterization of eigenvalues, that the functions \(y^D|\Gamma\) are eigenvalues of \(\Delta_T\), the Laplacian of \(\Gamma\), with eigenvalue \(\lambda\), for \(D = 1, \ldots, m\). We therefore have

\[
0 = (1/2)\Delta_T \sum_D (y^D|\Gamma)^2 = (1/2) \sum_D \Delta_T (y^D|\Gamma)^2 = \sum_D \left( (y^D|\Gamma) \Delta_T (y^D|\Gamma) + |\nabla_T (y^D|\Gamma)|^2 \right) = -\lambda \rho^2 + (n - 1);
\]

thus \(\lambda = (n - 1)/\rho^2\) and by T. Takahashi's theorem [12] \(\Gamma\) is minimal in \(S^{m-1}(\rho)\).

It is standard that if \(\Gamma\) is minimal in \(S^{m-1}(\rho)\) then the cone over \(\Gamma\) through the origin is minimal in \(\mathbb{R}^m\) (cf. e.g., [11, p. 97]). However \(\Omega\) and the cone over \(\Gamma\) are tangent along \(\Gamma\). By the Cauchy-Kowalewsky theorem [4, p. 39] \(\Omega\) is the cone over \(\Gamma\). Since \(\Omega\) is assumed to have no singularities, the cone must therefore be an \(n\)-dimensional subspace of \(\mathbb{R}^m\). Thus Theorem 2 is proven.

Remark. The differentiability of \(\Omega\) is crucial. Indeed given any minimal
submanifold $\Gamma$ of $S^{n-1}_{(\rho)}$, let $\Omega$ be the cone over $\Gamma$ through the origin of $R^m$. One easily sees that equality is obtained in (1).

**References**


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