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# ON A. HURWITZ' METHOD IN ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We show that if $M$ is complete simply connected with nonpositive sectional curvatures, $\Omega$ a minimal submanifold of $M$ with connected suitably oriented boundary $\Gamma$ then $\lambda^{1/2}V/A < (n-1)^{1/2}/n$ where $V$ is the volume of $\Omega$, $A$ the volume of $\Gamma$, $\lambda$ the first nonzero eigenvalue of the Laplacian of $\Gamma$, and $n$ ($\geqslant 2$) is the dimension of $\Omega$.

In this note we prove the following extension of A. Hurwitz' [7] argument in his proof of the classical isoperimetric inequality.

THEOREM 1. *Let $M$ be an $m$-dimensional complete simply connected Riemannian manifold all of whose sectional curvatures are nonpositive. Let $\Omega$ be an $n$-dimensional, $n \geqslant 2$, submanifold of $M$ with suitably oriented and connected boundary $\Gamma$. If $n < m$ assume $\Omega$ is minimal in $M$, i.e., its mean curvature vector vanishes identically. Let $V$ denote the $n$-volume of $\Omega$, $A$ the $(n-1)$-volume of $\Gamma$, and $\lambda$ the first nonzero eigenvalue of the Laplacian acting on functions on $\Gamma$. Then*

$$\sqrt{\lambda}\, V/A \leqslant \sqrt{n-1}\,/n. \tag{1}$$

Clearly the inequality (1) is sharp, since any $n$-disk in $R^m$ will yield equality in (1). We shall present a characterization of equality in (1) for two cases only.

THEOREM 2. *If $n = m$ then equality in* (1) *implies that $\Omega$ is isometric to a disk in $R^n$ endowed with the usual flat metric.*

*If $n < m$ and $M$ is $R^m$ endowed with the usual flat metric then equality in* (1) *implies that $\Omega$ is the intersection of a disk in $R^m$ with an $n$-dimensional affine space.*

In particular, Theorems 1 and 2 recapture T. Carleman's theorem [2] (also cf. [3]), that for a minimal surface $\Omega$ in $R^3$ with boundary $\Gamma$ consisting of a smooth Jordan curve, we have the inequality

$$L^2 - 4\pi A \geqslant 0 \tag{2}$$

where, just for the moment, $A$ is the area of $\Omega$ and $L$ the length of $\Gamma$. The

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point is that since $\Gamma$ is 1-dimensional Wirtinger's inequality implies $\lambda = 4\pi^2/L^2$. Equality in (2) is achieved if and only if $\Omega$ is a flat disk in $R^3$.

We note that the first to free Carleman's result and its generalizations from considerations of complex function theory was W. T. Reid **[9]**, who proved the result in 3-space *via* J. H. Jellet's formula (cf. (4) below), and C. C. Hsiung **[6]** who generalized Reid's argument to surfaces in $n$-space. We wish to thank R. Osserman for helpful conversations, guesses, and references. We mention R. C. Reilly [**10**, esp. Corollary 1] has also obtained results using A. Hurwitz' method as the starting point. Finally we refer the reader to the inequalities of D. Hoffman and J. Spruck **[5]** which, although not sharp, relate the volume and area alone.

**1. The inequality.** The manifolds $M$, $\Omega$, $\Gamma$ will be $C^\infty$, as will be the Riemannian metric under consideration. In general we will use *differentiable* for $C^\infty$.

We require some notation. Let $TM$ denote the tangent bundle of $M$, and let $\nabla$ denote covariant differentiation in $M$. For tangent vectors $\xi$, $\eta$ in the same fiber in $TM$ we denote their inner product by $\langle \xi, \eta \rangle$ and the associated norm of $\xi$, by $|\xi|$.

For an open set $U$ in $\Omega$ with coordinates chart $x$: $U \to R^n$ we let $\{\partial_1^x, \ldots, \partial_n^x\}$ denote the natural basis of tangent spaces to $\Omega$ at points of $U$ associated with $x$. We set

$$h_{jk} = \langle \partial_j^x, \partial_k^x \rangle, \quad \mathcal{H} = (h_{jk}), \quad \mathcal{H}^{-1} = (h^{jk}),$$

$j, k = 1, \ldots, n$. The mean curvature vector of $\Omega$ in $M$ will be denoted by $H$ (if $n = m$ then $H = 0$ by definition), $dV$ will denote the volume element of $\Omega$, and $dA$ that of $\Gamma$.

For a differentiable vector field $\xi$: $\bar{\Omega} \to TM$ ($\bar{\Omega} = \Omega \cup \Gamma$) on $\bar{\Omega}$ we let $\xi^T$ denote the projection of $\xi$ onto $T\bar{\Omega}$. For any open neighborhood $U$ in $\Omega$ with coordinate chart $x$ as above we define the *divergence of* $\xi$, $\operatorname{div}_\Omega \xi$, by

$$\operatorname{div}_\Omega \xi = \sum_{j,k=1}^{n} h^{jk} \langle \nabla \partial_j^x \xi, \partial_k^x \rangle.$$

It is standard that the above definition is independent of the coordinate chart on $\Omega$. Furthermore the infinitesimal version of the first variation of area reads as [**5**, p. 719]

$$\operatorname{div}_\Omega \xi^T = \operatorname{div}_\Omega \xi + \langle \xi, H \rangle. \tag{3}$$

Thus if $\nu$ denotes the outward normal vector field of $\Gamma$ with respect to $\Omega$ then

$$\int\int_\Omega \{\operatorname{div}_\Omega \xi + \langle \xi, H \rangle\} dV = \int_\Gamma \langle \xi, \nu \rangle \, dA. \tag{4}$$

We note that if $M$ is $R^n$ with the flat metric and $X$ is the position vector of points in $R^m$ suitably identified with a tangent vector field on $R^m$ then for $\xi = X|\Omega$ we have $\operatorname{div}_\Omega \xi = n$ on all of $\Omega$ and the resulting formula (4) is that of J. H. Jellet **[8]**.

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Next we let $p \in M$, $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $M_p$ and $y$: $M \to R^m$ the Riemannian normal coordinates on $M$ determined by $(p; e_1, \ldots, e_m)$. Our assumptions concerning $M$ imply by the Hadamard-Cartan theorem that $y$ is indeed defined on all of $M$ and is a diffeomorphism of $M$ onto $R^m$. It is standard that geodesics emanating from $p$ map onto rays emanating from the origin of $R^m$.

LEMMA 1. *We may choose $p$, $\{e_1, \ldots, e_m\}$ so that the respective coordinate functions $y^D$: $M \to R$, $D = 1, 2, \ldots, m$, of $y$: $M \to R^m$ satisfy*

$$\int_\Gamma y^D \, dA = 0. \tag{5}$$

PROOF. Our argument is an easy adaptation of one given by H. F. Weinberger [**13**, p. 635].

Parallel translate the frame $\{e_1, \ldots, e_m\}$ along every geodesic emanating from $p$ and thereby obtain a differentiable orthonormal frame field $\{E_1, \ldots, E_m\}$ on $M$. Let $y_q$: $M \to R^m$ denote the Riemann normal coordinates of $M$ determined by $\{E_1, \ldots, E_m\}$ at $q$, and let $(y_q)^D$, $D = 1, \ldots, m$ be the coordinate functions of $y_q$. Then

$$Y(q) = \sum_{D=1}^{m} \left\{ \int_\Gamma (y_q)^D \, dA \right\} E_D(q)$$

is a continuous vector field on $M$. If we restrict $Y$ to a geodesic disk $B$ containing $\Omega$ then the convexity of $B$ implies that on the boundary of $B$, $Y$ points into $B$. The Brouwer fixed point theorem then implies that $Y$ has zero on $B$.

So we may assume that $p$, $\{e_1, \ldots, e_m\}$ actually satisfies (5). Let $X$ be the vector field on $M$ given by

$$X = \sum_{D=1}^{m} y^D \partial_D^y$$

where $\{\partial_D^y, D = 1, \ldots, m\}$ is the natural basis of tangent spaces associated with the coordinate chart $y$. Of course if $M$ is $R^m$ with its usual flat metric then $X$ is naturally identified with the position vector. Next set $\xi = X|\Omega$, i.e. the restriction of $X$ to $\Omega$. Then standard arguments using the Rauch comparison theorem [**5**, p. 721] imply

$$n \leqslant \operatorname{div}_\Omega \xi. \tag{6}$$

Thus, if $H = 0$ on $\Omega$, then (4), (5), (6) imply

$$\begin{aligned} nV &\leqslant \int_\Gamma \langle \xi, \nu \rangle \, dA \leqslant \int_\Gamma |\xi| dA \\ &\leqslant A^{1/2} \left\{ \int_\Gamma |\xi|^2 \, dA \right\}^{1/2} = A^{1/2} \left\{ \int_\Gamma \sum_D (y^D)^2 \, dA \right\}^{1/2} \\ &\leqslant (A/\lambda)^{1/2} \left\{ \int_\Gamma \sum_D |\operatorname{grad}_\Gamma y^D|^2 \, dA \right\}^{1/2} \end{aligned}$$

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The expression $\operatorname{grad}_\Gamma y^D$ denotes the gradient of $(y^D|\Gamma)$ in $\Gamma$. The last inequality is, combined with (5), Lord Rayleigh's characterization of the first nonzero eigenvalue $\lambda$ of the Laplacian on $\Gamma$.

It remains to verify the estimate

$$\sum_D |\operatorname{grad}_\Gamma y^D|^2 \leqslant n - 1 \tag{7}$$

on all of $\Gamma$.

Let $q \in \Gamma$, $u$: $G \to R^{n-1}$ be a coordinate chart on $\Gamma$ about $q$, $\{\partial_\alpha^u, \alpha = 1, \ldots, n-1\}$ the natural basis of tangent spaces to $\Gamma$ at points of $G$ such that $\{\partial_1^u, \ldots, \partial_{n-1}^u\}$ is orthonormal at $q$. For $y$: $M \to R^m$ let $g_{AB} = \langle \partial_A^y, \partial_B^y \rangle$; then the Rauch comparison theorem implies that the eigenvalues of $(g_{AB})$ are all $\geqslant 1$. Thus at $q$

$$\sum_{D=1}^m |\operatorname{grad}_\Gamma y^D|^2 = \sum_{D=1}^m \sum_{\alpha=1}^{n-1} (\partial y^D/\partial u^\alpha)^2 = \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^m \frac{\partial y^C}{\partial u^\alpha} \delta_{CD} \frac{\partial y^D}{\partial u^\alpha}$$
$$\leqslant \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^m \frac{\partial y^C}{\partial u^\alpha} g_{CD} \frac{\partial y^D}{\partial u^\alpha} = \sum_{\alpha=1}^{n-1} \langle \partial_\alpha^u, \partial_\alpha^u \rangle = n-1,$$

and Theorem 1 is proven.

**2. The case of equality for** $m = n$. If $m = n$ and we have equality in (1) then $|\xi|$ is constant on $\Gamma$ from which one concludes that $\Gamma$ is a geodesic sphere bounding the geodesic disk $\Omega$. Furthermore $\operatorname{div}_\Omega \xi = n$ on all of $\Omega$. But this in turn implies by [**1**, pp. 253–257; 5, ibid.] that $\Omega$ is isometric to a disk in $R^n$.

**3. The case of equality for** $n < m$, $M = R^m$. Since $M$ is $R^m$ we identify $\xi$ at $q$ with the position vector at $q$, and note that equality in (1) implies that $\Gamma$ is contained in some sphere $S^{m-1}(\rho)$ of radius $\rho$ about the origin and that, furthermore, that the outward normal, $\nu$, of $\Gamma$ with respect to $\Omega$ is $\rho^{-1}\xi$.

Next we note that equality in (1) implies, by the Rayleigh characterization of eigenvalues, that the functions $y^D|\Gamma$ are eigenvalues of $\Delta_\Gamma$, the Laplacian of $\Gamma$, with eigenvalue $\lambda$, for $D = 1, \ldots, m$. We therefore have

$$0 = (1/2)\Delta_\Gamma \sum_D (y^D|\Gamma)^2 = (1/2) \sum_D \Delta_\Gamma (y^D|\Gamma)^2$$
$$= \sum_D \left\{ (y^D|\Gamma)\Delta_\Gamma(y^D|\Gamma) + \left|\operatorname{grad}_\Gamma (y^D|\Gamma)\right|^2 \right\} = -\lambda\rho^2 + (n-1);$$

thus $\lambda = (n-1)/\rho^2$ and by T. Takahashi's theorem [**12**] $\Gamma$ is minimal in $S^{m-1}(\rho)$.

It is standard that if $\Gamma$ is minimal in $S^{m-1}(\rho)$ then the cone over $\Gamma$ through the origin is minimal in $R^m$ (cf. e.g., [**11**, p. 97]). However $\Omega$ and the cone over $\Gamma$ are tangent along $\Gamma$. By the Cauchy-Kowalewsky theorem [**4**, p. 39] $\Omega$ *is* the cone over $\Gamma$. Since $\Omega$ is assumed to have no singularities, the cone must therefore be an $n$-dimensional subspace of $R^m$. Thus Theorem 2 is proven.

Remark. The differentiability of $\Omega$ is crucial. Indeed given *any* minimal

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submanifold $\Gamma$ of $S^{n-1}_{(\rho)}$, let $\Omega$ be the cone over $\Gamma$ through the origin of $R^m$. One easily sees that equality is obtained in (1).

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