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ON A. HURWITZ' METHOD IN ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We show that if M is complete simply connected with nonpositive sectional curvatures, Ω a minimal submanifold of M with connected suitably oriented boundary Γ then $\lambda^{1/2}V/A \leqslant (n-1)^{1/2}/n$ where V is the volume of Ω, A the volume of Γ, λ the first nonzero eigenvalue of the Laplacian of Γ, and n ($\geqslant 2$) is the dimension of Ω.

In this note we prove the following extension of A. Hurwitz' [7] argument in his proof of the classical isoperimetric inequality.

THEOREM 1. *Let M be an m-dimensional complete simply connected Riemannian manifold all of whose sectional curvatures are nonpositive. Let Ω be an n-dimensional, $n \geqslant 2$, submanifold of M with suitably oriented and connected boundary Γ. If $n < m$ assume Ω is minimal in M, i.e., its mean curvature vector vanishes identically. Let V denote the n-volume of Ω, A the $(n-1)$-volume of Γ, and λ the first nonzero eigenvalue of the Laplacian acting on functions on Γ. Then*

$$\sqrt{\lambda}\, V/A \leqslant \sqrt{n-1}\,/n. \tag{1}$$

Clearly the inequality (1) is sharp, since any n-disk in R^m will yield equality in (1). We shall present a characterization of equality in (1) for two cases only.

THEOREM 2. *If $n = m$ then equality in* (1) *implies that Ω is isometric to a disk in R^n endowed with the usual flat metric.*

If $n < m$ and M is R^m endowed with the usual flat metric then equality in (1) *implies that Ω is the intersection of a disk in R^m with an n-dimensional affine space.*

In particular, Theorems 1 and 2 recapture T. Carleman's theorem [2] (also cf. [3]), that for a minimal surface Ω in R^3 with boundary Γ consisting of a smooth Jordan curve, we have the inequality

$$L^2 - 4\pi A \geqslant 0 \tag{2}$$

where, just for the moment, A is the area of Ω and L the length of Γ. The

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point is that since Γ is 1-dimensional Wirtinger's inequality implies $\lambda = 4\pi^2/L^2$. Equality in (2) is achieved if and only if Ω is a flat disk in R^3.

We note that the first to free Carleman's result and its generalizations from considerations of complex function theory was W. T. Reid **[9]**, who proved the result in 3-space *via* J. H. Jellet's formula (cf. (4) below), and C. C. Hsiung **[6]** who generalized Reid's argument to surfaces in n-space. We wish to thank R. Osserman for helpful conversations, guesses, and references. We mention R. C. Reilly [**10**, esp. Corollary 1] has also obtained results using A. Hurwitz' method as the starting point. Finally we refer the reader to the inequalities of D. Hoffman and J. Spruck **[5]** which, although not sharp, relate the volume and area alone.

1. The inequality. The manifolds M, Ω, Γ will be C^∞, as will be the Riemannian metric under consideration. In general we will use *differentiable* for C^∞.

We require some notation. Let TM denote the tangent bundle of M, and let ∇ denote covariant differentiation in M. For tangent vectors ξ, η in the same fiber in TM we denote their inner product by $\langle \xi, \eta \rangle$ and the associated norm of ξ, by $|\xi|$.

For an open set U in Ω with coordinates chart x: $U \to R^n$ we let $\{\partial_1^x, \ldots, \partial_n^x\}$ denote the natural basis of tangent spaces to Ω at points of U associated with x. We set

$$h_{jk} = \langle \partial_j^x, \partial_k^x \rangle, \quad \mathcal{H} = (h_{jk}), \quad \mathcal{H}^{-1} = (h^{jk}),$$

$j, k = 1, \ldots, n$. The mean curvature vector of Ω in M will be denoted by H (if $n = m$ then $H = 0$ by definition), dV will denote the volume element of Ω, and dA that of Γ.

For a differentiable vector field ξ: $\bar{\Omega} \to TM$ ($\bar{\Omega} = \Omega \cup \Gamma$) on $\bar{\Omega}$ we let ξ^T denote the projection of ξ onto $T\bar{\Omega}$. For any open neighborhood U in Ω with coordinate chart x as above we define the *divergence of* ξ, $\operatorname{div}_\Omega \xi$, by

$$\operatorname{div}_\Omega \xi = \sum_{j,k=1}^{n} h^{jk} \langle \nabla \partial_j^x \xi, \partial_k^x \rangle.$$

It is standard that the above definition is independent of the coordinate chart on Ω. Furthermore the infinitesimal version of the first variation of area reads as [**5**, p. 719]

$$\operatorname{div}_\Omega \xi^T = \operatorname{div}_\Omega \xi + \langle \xi, H \rangle. \tag{3}$$

Thus if ν denotes the outward normal vector field of Γ with respect to Ω then

$$\int\int_\Omega \{\operatorname{div}_\Omega \xi + \langle \xi, H \rangle\} dV = \int_\Gamma \langle \xi, \nu \rangle \, dA. \tag{4}$$

We note that if M is R^n with the flat metric and X is the position vector of points in R^m suitably identified with a tangent vector field on R^m then for $\xi = X|\Omega$ we have $\operatorname{div}_\Omega \xi = n$ on all of Ω and the resulting formula (4) is that of J. H. Jellet **[8]**.

Next we let $p \in M$, $\{e_1, \ldots, e_m\}$ be an orthonormal basis of M_p and y: $M \to R^m$ the Riemannian normal coordinates on M determined by $(p; e_1, \ldots, e_m)$. Our assumptions concerning M imply by the Hadamard-Cartan theorem that y is indeed defined on all of M and is a diffeomorphism of M onto R^m. It is standard that geodesics emanating from p map onto rays emanating from the origin of R^m.

LEMMA 1. *We may choose p, $\{e_1, \ldots, e_m\}$ so that the respective coordinate functions y^D: $M \to R$, $D = 1, 2, \ldots, m$, of y: $M \to R^m$ satisfy*

$$\int_\Gamma y^D \, dA = 0. \tag{5}$$

PROOF. Our argument is an easy adaptation of one given by H. F. Weinberger [**13**, p. 635].

Parallel translate the frame $\{e_1, \ldots, e_m\}$ along every geodesic emanating from p and thereby obtain a differentiable orthonormal frame field $\{E_1, \ldots, E_m\}$ on M. Let y_q: $M \to R^m$ denote the Riemann normal coordinates of M determined by $\{E_1, \ldots, E_m\}$ at q, and let $(y_q)^D$, $D = 1, \ldots, m$ be the coordinate functions of y_q. Then

$$Y(q) = \sum_{D=1}^{m} \left\{ \int_\Gamma (y_q)^D \, dA \right\} E_D(q)$$

is a continuous vector field on M. If we restrict Y to a geodesic disk B containing Ω then the convexity of B implies that on the boundary of B, Y points into B. The Brouwer fixed point theorem then implies that Y has zero on B.

So we may assume that p, $\{e_1, \ldots, e_m\}$ actually satisfies (5). Let X be the vector field on M given by

$$X = \sum_{D=1}^{m} y^D \partial_D^y$$

where $\{\partial_D^y, D = 1, \ldots, m\}$ is the natural basis of tangent spaces associated with the coordinate chart y. Of course if M is R^m with its usual flat metric then X is naturally identified with the position vector. Next set $\xi = X|\Omega$, i.e. the restriction of X to Ω. Then standard arguments using the Rauch comparison theorem [**5**, p. 721] imply

$$n \leqslant \operatorname{div}_\Omega \xi. \tag{6}$$

Thus, if $H = 0$ on Ω, then (4), (5), (6) imply

$$\begin{aligned} nV &\leqslant \int_\Gamma \langle \xi, \nu \rangle \, dA \leqslant \int_\Gamma |\xi| dA \\ &\leqslant A^{1/2} \left\{ \int_\Gamma |\xi|^2 \, dA \right\}^{1/2} = A^{1/2} \left\{ \int_\Gamma \sum_D (y^D)^2 \, dA \right\}^{1/2} \\ &\leqslant (A/\lambda)^{1/2} \left\{ \int_\Gamma \sum_D |\operatorname{grad}_\Gamma y^D|^2 \, dA \right\}^{1/2} \end{aligned}$$

The expression $\operatorname{grad}_\Gamma y^D$ denotes the gradient of $(y^D|\Gamma)$ in Γ. The last inequality is, combined with (5), Lord Rayleigh's characterization of the first nonzero eigenvalue λ of the Laplacian on Γ.

It remains to verify the estimate

$$\sum_D |\operatorname{grad}_\Gamma y^D|^2 \leqslant n - 1 \tag{7}$$

on all of Γ.

Let $q \in \Gamma$, u: $G \to R^{n-1}$ be a coordinate chart on Γ about q, $\{\partial_\alpha^u, \alpha = 1, \ldots, n-1\}$ the natural basis of tangent spaces to Γ at points of G such that $\{\partial_1^u, \ldots, \partial_{n-1}^u\}$ is orthonormal at q. For y: $M \to R^m$ let $g_{AB} = \langle \partial_A^y, \partial_B^y \rangle$; then the Rauch comparison theorem implies that the eigenvalues of (g_{AB}) are all $\geqslant 1$. Thus at q

$$\sum_{D=1}^{m} |\operatorname{grad}_\Gamma y^D|^2 = \sum_{D=1}^{m} \sum_{\alpha=1}^{n-1} (\partial y^D / \partial u^\alpha)^2 = \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^{m} \frac{\partial y^C}{\partial u^\alpha} \delta_{CD} \frac{\partial y^D}{\partial u^\alpha}$$

$$\leqslant \sum_{\alpha=1}^{n-1} \sum_{C,D=1}^{m} \frac{\partial y^C}{\partial u^\alpha} g_{CD} \frac{\partial y^D}{\partial u^\alpha} = \sum_{\alpha=1}^{n-1} \langle \partial_\alpha^u, \partial_\alpha^u \rangle = n - 1,$$

and Theorem 1 is proven.

2. The case of equality for $m = n$. If $m = n$ and we have equality in (1) then $|\xi|$ is constant on Γ from which one concludes that Γ is a geodesic sphere bounding the geodesic disk Ω. Furthermore $\operatorname{div}_\Omega \xi = n$ on all of Ω. But this in turn implies by [**1**, pp. 253–257; 5, ibid.] that Ω is isometric to a disk in R^n.

3. The case of equality for $n < m$, $M = R^m$. Since M is R^m we identify ξ at q with the position vector at q, and note that equality in (1) implies that Γ is contained in some sphere $S^{m-1}(\rho)$ of radius ρ about the origin and that, furthermore, that the outward normal, ν, of Γ with respect to Ω is $\rho^{-1}\xi$.

Next we note that equality in (1) implies, by the Rayleigh characterization of eigenvalues, that the functions $y^D|\Gamma$ are eigenvalues of Δ_Γ, the Laplacian of Γ, with eigenvalue λ, for $D = 1, \ldots, m$. We therefore have

$$0 = (1/2)\Delta_\Gamma \sum_D (y^D|\Gamma)^2 = (1/2) \sum_D \Delta_\Gamma (y^D|\Gamma)^2$$

$$= \sum_D \left\{ (y^D|\Gamma)\Delta_\Gamma(y^D|\Gamma) + |\operatorname{grad}_\Gamma (y^D|\Gamma)|^2 \right\} = -\lambda\rho^2 + (n-1);$$

thus $\lambda = (n-1)/\rho^2$ and by T. Takahashi's theorem [**12**] Γ is minimal in $S^{m-1}(\rho)$.

It is standard that if Γ is minimal in $S^{m-1}(\rho)$ then the cone over Γ through the origin is minimal in R^m (cf. e.g., [**11**, p. 97]). However Ω and the cone over Γ are tangent along Γ. By the Cauchy-Kowalewsky theorem [**4**, p. 39] Ω *is* the cone over Γ. Since Ω is assumed to have no singularities, the cone must therefore be an n-dimensional subspace of R^m. Thus Theorem 2 is proven.

REMARK. The differentiability of Ω is crucial. Indeed given *any* minimal submanifold Γ of $S^{n-1}_{(\rho)}$, let Ω be the cone over Γ through the origin of R^m. One easily sees that equality is obtained in (1).

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