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RESOLUTIONS OF H-CLOSED SPACES

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ABSTRACT. It is shown that every Hausdorff space X is the perfect, irreducible, continuous image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) \leq w(X)$, where $w(\tilde{X})$ represents the weight of X, and if X is H-closed then $\tilde{X}$ is also H-closed. A corollary of this result is that if f: $X \to Y$ is a continuous map of the H-closed space X onto the semi-regular Hausdorff space Y, then $w(Y) \leq w(X)$.

A Hausdorff space is *H-closed* if it is closed in every Hausdorff space in which it is embedded. A space is *extremally disconnected* if the closure of every open set is open. An extremally disconnected Hausdorff space $\tilde{X}$ will be called a *continuous* or *θ-continuous resolution* of a space X if there is a function f: $\tilde{X} \to X$ which is continuous or θ-continuous respectively and also perfect, irreducible, and onto. (See **[6]** for the definition of θ-continuous.) $w(X)$ will denote the weight of a space X.

It is known that: (1) (**[9]** and **[11]**) every regular T_1 space X has a continuous resolution which is compact whenever X is; (2) **[6]** every Hausdorff space has a θ-continuous resolution which is compact whenever X is H-closed; (3) **[8]** every Hausdorff space is the continuous image of an extremally disconnected Hausdorff space which is H-closed whenever X is. It follows from the result of V. Ponomarev **[9**, Theorem 1] that for a space X, the weight of the resolution in (1), (2), or (3) above is less than or equal to $2^{2^{w(X)}}$. Further, Ponomarev proved that this is the best possible estimate.

The purpose of this note is to show that by a slight weakening of the extremal disconnectedness of the pre-image, it is possible to obtain a much better control on its weight. A corollary of the main result is that if g: $X \to Y$ is a continuous map of the H-closed space X onto the semiregular Hausdorff space Y, then $w(Y) \leq w(X)$. We note that instead of using spaces of open ultrafilters, as in **[5]**, **[6]**, **[8]**, and **[11]**, or the methods of **[9]**, the proof of the main theorem uses inverse limit techniques similar to those employed by Friedler and Pettey **[4]** and Dickman and Vinson **[2]**. Notation and terminology will follow **[4]**.

DEFINITIONS. A function f: $X \to Y$ is *semi-open* [7] if $\operatorname{int} f(U)$ is nonempty whenever U is a nonempty open subset of X. A function f: $X \to Y$ is *irreducible* if f is onto and for no proper closed subset F of X is $f(F) = Y$. Note that "irreducible" has a slightly different meaning in **[8]**.

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LEMMA 1. *Let f: $X \to Y$ be a closed, irreducible, and onto map. Then f is semi-open.*

PROOF. Let U be a nonempty open subset of X. Then $X - U$ is a proper closed subset, hence $f(X - U) \neq Y$ and $Y - f(X - U)$ is a nonempty open subset of $f(U)$.

LEMMA 2. *Let V be an open subset of the Hausdorff space Y, let X be the disjoint union of $\mathrm{Cl}_Y(Y - \mathrm{Cl}_Y V)$ and $\mathrm{Cl}_Y V$, and let f be the natural map of X onto Y. Then f is perfect, irreducible, semi-open, continuous and onto, and X is H-closed whenever Y is H-closed.*

PROOF. It is easily verified that f is perfect, irreducible, continuous and onto and Lemma 1 implies that f is semi-open. If Y is H-closed, both $\mathrm{Cl}_Y V$ and $\mathrm{Cl}_Y(Y - \mathrm{Cl}_Y V)$ are the closures of open subsets of Y, and hence H-closed [**1**, Theorem 3.3]. Then X, as the disjoint union of two H-closed spaces, is also H-closed.

In the sequel we may refer to V as a subset of X as well as of Y.

LEMMA 3. *If f: $X \to Y$ is a semi-open, onto, continuous map and $\mathrm{Cl}\, V$ is open in Y, then $f^{-1}(\mathrm{Cl}\, V) = \mathrm{Cl}\, f^{-1}V$.*

PROOF. The inclusion $\mathrm{Cl}\, f^{-1}V \subset f^{-1}(\mathrm{Cl}\, V)$ follows from continuity, so let $x \in f^{-1}(\mathrm{Cl}\, V)$ and let U be a neighborhood of x. Then $f(x) \in \mathrm{Cl}\, V$, which is open. By continuity, x has a neighborhood $U' \subset U$ such that $f(U') \subset \mathrm{Cl}\, V$. Since f is semi-open, $\mathrm{int}\, f(U') \neq \varnothing$. But if $\mathrm{int}\, f(U') \cap \mathrm{Cl}\, V \neq \varnothing$, then $\mathrm{int}\, f(U') \cap V \neq \varnothing$. If $y \in \mathrm{int}\, f(U') \cap V$ and $z \in f^{-1}(y) \cap U'$, then $z \in U \cap f^{-1}(V)$ so that $x \in \mathrm{Cl}\, f^{-1}(V)$.

COROLLARY. *If f: $X \to Y$ is semi-open, continuous and onto and Y is extremally disconnected, then $f^{-1}(\mathrm{Cl}\, V) = \mathrm{Cl}\, f^{-1}(V)$ for every open subset V of Y.*

Maps such that $f^{-1}(\mathrm{Cl}\, V) = \mathrm{Cl}\, f^{-1}(V)$ for every open subset V were studied in [**4**].

The proof of the next lemma is routine.

LEMMA 4. *Let $\{X_\alpha; f^\alpha_\beta\}$ be an inverse system with each bonding map closed, irreducible, and onto and each projection closed and onto. Then each projection is irreducible (and hence semi-open by Lemma 1).*

THEOREM. *Every Hausdorff space X is the continuous, perfect, irreducible, semi-open image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) \leqslant w(X)$, and if X is H-closed then $\tilde{X}$ is H-closed.*

PROOF. Let $\mathfrak{T}$ be the initial ordinal of cardinality $w(X)$ and by [**4**, Lemma 2.1] let σ be a one-to-one function from $[0, \mathfrak{T}) \times [0, \mathfrak{T})$ onto $[1, \mathfrak{T})$ such that for each pair (α, β), $\sigma(\alpha, \beta) > \alpha$. We define an inverse system $\{X_\beta; f^\beta_\alpha\}$, $\alpha \leqslant \beta < \mathfrak{T}$ such that for each $\beta < \mathfrak{T}$, $w(X_\beta) \leqslant w(X)$. For each $\beta < \mathfrak{T}$ we shall pick a basis $\{U_{\beta\gamma}: \gamma < \mathfrak{T}\}$ for X_β and shall assume that f_β^β is the identity mapping on X_β.

Let $X_0 = X$ and choose a base $\{U_{0\gamma}: \gamma \in [0, \mathfrak{T})\}$ for X_0. For each $\beta < \mathfrak{T}$, let Z_β denote the inverse limit of the system $\{X_\alpha; f_\gamma^\alpha\}$, $\gamma \leqslant \alpha \leqslant \beta$, and for each $\alpha < \beta$, let ϕ_α^β denote the projection from Z_β into X_α. If $\sigma^{-1}(\beta) = (\lambda, \eta)$, then $\beta > \lambda$ so $U_{\lambda\eta}$ has already been defined. Let $V_\beta = (\phi_\lambda^\beta)^{-1}(U_{\lambda\eta})$ and let X_β be the disjoint union of $\mathrm{Cl}_{Z_\beta} V$ and $\mathrm{Cl}_{Z_\beta}(Z_\beta - \mathrm{Cl}_{Z_\beta} V_\beta)$. Let p_β be the projection of X_β onto Z_β and for each α less than β, let $f_\alpha^\beta = \phi_\alpha^\beta \circ p_\beta$. Clearly, for $0 \leqslant \alpha \leqslant \gamma \leqslant \beta$, $f_\alpha^\beta = f_\alpha^\gamma f_\gamma^\beta$. From [**4**, Lemma 2.2] it follows that $w(X_\beta) \leqslant w(X)$, so choose a base $\{U_{\beta\gamma}: \gamma \in [0, \mathfrak{T})\}$ for X_β.

Let $\tilde{X}$ denote the inverse limit of $\{X_\beta; f_\alpha^\beta\}$, $\alpha \leqslant \beta < \mathfrak{T}$. By [**4**, Lemma 2.2], $w(\tilde{X}) \leqslant w(X)$.

If, for each $\alpha < \beta$, each bonding map f_γ^α $(\alpha \geqslant \gamma)$ is perfect, irreducible, semi-open, onto, and continuous, then [**4**, Lemma 2.5] and Lemma 4 imply that Z_β is nonempty and each projection $\phi_\alpha^\beta: Z_\beta \to X_\alpha$ is perfect, irreducible, semi-open, onto and continuous. Again, by [**4**, Lemma 2.5] and Lemmas 2 and 4, $\tilde{X}$ is nonempty and each projection $f_\beta: \tilde{X} \to X_\beta$ is perfect, irreducible, semi-open, onto and continuous. Let $f = f_0$ be the map from $\tilde{X}$ onto X. As a subspace of a product of Hausdorff spaces, $\tilde{X}$ is Hausdorff.

Let U be a neighborhood of $x \in \tilde{X}$. It follows from the construction that $x \in f_\beta^{-1}(V_\beta) \subset U$ for some $\beta < \mathfrak{T}$. Since each projection is semi-open, onto, and continuous and $\mathrm{Cl}_{X_\beta}(V_\beta)$ is open in X_β, Lemma 3 implies that $\mathrm{Cl}_{\tilde{X}} f_\beta^{-1}(V_\beta) = f_\beta^{-1}(\mathrm{Cl}_{X_\beta} V_\beta)$. Hence, $\{f_\beta^{-1}(V_\beta): \beta < \mathfrak{T}\}$ is a basis for $\tilde{X}$ and every member has an open closure.

Finally, if each X_α $(\alpha < \beta)$ is H-closed, then since each f_γ^α and each ϕ_α^β is semi-open and onto, it follows from [**7**, Theorem 3.7] that each Z_β is H-closed and from Lemma 2 that each X_β is H-closed. Again by [**7**, Theorem 3.7], $\tilde{X}$ is H-closed.

Corollary. *Let $g: X \to Y$ be a continuous map of an H-closed space X onto a semiregular, Hausdorff space Y. Then $w(Y) \leqslant w(X)$.*

Proof. Let $\tilde{X}$ be the space constructed in Theorem 1 and let f be the map from $\tilde{X}$ onto X. By Theorem 1, $\tilde{X}$ has a base of cardinality $w(X)$. Consider $F = g \circ f: \tilde{X} \to Y$. Since $\tilde{X}$ is H-closed, $F^{-1}(y)$ is H-closed for every $y \in Y$ [**12**, Theorem 1.4]. Let $\{U_\alpha\}$ be a base for $\tilde{X}$ of cardinality $w(X)$ and such that each member has an open closure. If $\{V_\alpha\}$ is the collection of all finite unions of closures of members of $\{U_\alpha\}$, then each V_α, and hence each $\tilde{X} - V_\alpha$, is clopen and so H-closed. Thus, $F(\tilde{X} - V_\alpha)$ is closed in Y for each α. For each α let $W_\alpha = Y - F(\tilde{X} - V_\alpha)$ and let $y \in W$, an open subset of Y. If V is a regularly open neighborhood of y contained in W, then $F^{-1}(y) \subset F^{-1}(V)$, so that by the covering characterization of H-closed spaces, $F^{-1}(y) \subset V_\alpha \subset \mathrm{Cl}\, F^{-1}(V)$ for some α. It follows that $y \in W_\alpha = \mathrm{int}\, W_\alpha \subset \mathrm{int}\,\mathrm{Cl}\, V = V \subset W$, and hence $\{W_\alpha\}$ is a base for Y of cardinality $w(X)$. But then $w(Y) \leqslant w(X)$.

REMARKS. It follows from [**9**, Theorem 1] that the absolute of the closed unit interval is not second countable. (See [**6**] for the definition of absolute.) This fact, together with [**8**, Theorem 3.4] and the preceeding corollary imply the following.

PROPOSITION. *No second countable, extremally disconnected H-closed space can be mapped onto the closed unit interval by an irreducible, closed continuous mapping.*

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