RESOLUTIONS OF $H$-CLOSED SPACES

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ABSTRACT. It is shown that every Hausdorff space $X$ is the perfect, irreducible, continuous image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) < w(X)$, where $w(\tilde{X})$ represents the weight of $X$, and if $X$ is $H$-closed then $\tilde{X}$ is also $H$-closed. A corollary of this result is that if $f: X \to Y$ is a continuous map of the $H$-closed space $X$ onto the semi-regular Hausdorff space $Y$, then $w(Y) < w(X)$.

A Hausdorff space is $H$-closed if it is closed in every Hausdorff space in which it is embedded. A space is extremally disconnected if the closure of every open set is open. An extremally disconnected Hausdorff space $\tilde{X}$ will be called a continuous or $\theta$-continuous resolution of a space $X$ if there is a function $f: \tilde{X} \to X$ which is continuous or $\theta$-continuous respectively and also perfect, irreducible, and onto. (See [6] for the definition of $\theta$-continuous.) $w(X)$ will denote the weight of a space $X$.

It is known that: (1) ([9] and [11]) every regular $T_1$ space $X$ has a continuous resolution which is compact whenever $X$ is; (2) [6] every Hausdorff space has a $\theta$-continuous resolution which is compact whenever $X$ is $H$-closed; (3) [8] every Hausdorff space is the continuous image of an extremally disconnected Hausdorff space which is $H$-closed whenever $X$ is. It follows from the result of V. Ponomarev [9, Theorem 1] that for a space $X$, the weight of the resolution in (1), (2), or (3) above is less than or equal to $2^{2w(X)}$. Further, Ponomarev proved that this is the best possible estimate.

The purpose of this note is to show that by a slight weakening of the extremal disconnectedness of the pre-image, it is possible to obtain a much better control on its weight. A corollary of the main result is that if $g: X \to Y$ is a continuous map of the $H$-closed space $X$ onto the semiregular Hausdorff space $Y$, then $w(Y) < w(X)$.

Definitions. A function $f: X \to Y$ is semi-open [7] if $\text{int} f(U)$ is nonempty whenever $U$ is a nonempty open subset of $X$. A function $f: X \to Y$ is irreducible if $f$ is onto and for no proper closed subset $F$ of $X$ is $f(F) = Y$. Note that "irreducible" has a slightly different meaning in [8].

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Lemma 1. Let $f : X \to Y$ be a closed, irreducible, and onto map. Then $f$ is semi-open.

Proof. Let $U$ be a nonempty open subset of $X$. Then $X - U$ is a proper closed subset, hence $f(X - U) \neq Y$ and $Y - f(X - U)$ is a nonempty open subset of $f(U)$.

Lemma 2. Let $V$ be an open subset of the Hausdorff space $Y$, let $X$ be the disjoint union of $\text{Cl}_Y(Y - \text{Cl}_Y V)$ and $\text{Cl}_Y V$, and let $f$ be the natural map of $X$ onto $Y$. Then $f$ is perfect, irreducible, semi-open, continuous and onto, and $X$ is $H$-closed whenever $Y$ is $H$-closed.

Proof. It is easily verified that $f$ is perfect, irreducible, continuous and onto and Lemma 1 implies that $f$ is semi-open. If $Y$ is $H$-closed, both $\text{Cl}_Y V$ and $\text{Cl}_Y(Y - \text{Cl}_Y V)$ are the closures of open subsets of $Y$, and hence $H$-closed [1, Theorem 3.3]. Then $X$, as the disjoint union of two $H$-closed spaces, is also $H$-closed.

In the sequel we may refer to $V$ as a subset of $X$ as well as of $Y$.

Lemma 3. If $f : X \to Y$ is a semi-open, onto, continuous map and $\text{Cl}_Y V$ is open in $Y$, then $f^{-1}(\text{Cl}_Y V) = \text{Cl}_X f^{-1}(V)$.

Proof. The inclusion $\text{Cl}_X f^{-1}(\text{Cl}_Y V) \subset f^{-1}(\text{Cl}_Y V)$ follows from continuity, so let $x \in f^{-1}(\text{Cl}_Y V)$ and let $U$ be a neighborhood of $x$. Then $f(x) \in \text{Cl}_Y V$, which is open. By continuity, $x$ has a neighborhood $U' \subset U$ such that $f(U') \subset \text{Cl}_Y V$.

Since $f$ is semi-open, $\text{int} f(U') \neq \emptyset$. But if $\text{int} f(U') \cap \text{Cl}_Y V \neq \emptyset$, then $\text{int} f(U') \cap V \neq \emptyset$. If $y \in \text{int} f(U') \cap V$ and $z \in f^{-1}(y) \cap U'$, then $z \in U \cap f^{-1}(V)$ so that $x \in \text{Cl}_X f^{-1}(V)$.

Corollary. If $f : X \to Y$ is semi-open, continuous and onto and $Y$ is extremally disconnected, then $f^{-1}(\text{Cl}_Y V) = \text{Cl}_X f^{-1}(V)$ for every open subset $V$ of $Y$.

Maps such that $f^{-1}(\text{Cl}_Y V) = \text{Cl}_X f^{-1}(V)$ for every open subset $V$ were studied in [4].

The proof of the next lemma is routine.

Lemma 4. Let $\{X_\alpha; f_\beta^\alpha\}$ be an inverse system with each bonding map closed, irreducible, and onto and each projection closed and onto. Then each projection is irreducible (and hence semi-open by Lemma 1).

Theorem. Every Hausdorff space $X$ is the continuous, perfect, irreducible, semi-open image of a Hausdorff space $\bar{X}$ which has a basis with open closures. Further, $w(\bar{X}) < w(X)$, and if $X$ is $H$-closed then $\bar{X}$ is $H$-closed.

Proof. Let $\bar{\mathcal{T}}$ be the initial ordinal of cardinality $w(X)$ and by [4, Lemma 2.1] let $\sigma$ be a one-to-one function from $[0, \bar{\mathcal{T}}] \times [0, \bar{\mathcal{T}}]$ onto $[1, \bar{\mathcal{T}}]$ such that for each pair $(\alpha, \beta)$, $\sigma(\alpha, \beta) > \alpha$. We define an inverse system $\{X_\beta; f_\beta^\alpha\}$, $\alpha < \beta < \bar{\mathcal{T}}$ such that for each $\beta < \bar{\mathcal{T}}$, $w(X_\beta) < w(X)$. For each $\beta < \bar{\mathcal{T}}$ we
shall pick a basis \( \{ U_\gamma : \gamma < \bar{\gamma} \} \) for \( X_\beta \) and shall assume that \( f_\beta^\beta \) is the identity mapping on \( X_\beta \).

Let \( X_0 = X \) and choose a base \( \{ U_\gamma : \gamma \in [0, \bar{\gamma}) \} \) for \( X_0 \). For each \( \beta < \bar{\gamma} \), let \( Z_\beta \) denote the inverse limit of the system \( \{ X_\alpha ; /_\alpha \} \), \( \gamma < \alpha < \beta \), and for each \( \alpha < \beta \), let \( \phi_\alpha^\beta \) denote the projection from \( Z_\beta \) into \( X_\alpha \). If \( \alpha^{-1}(\beta) = (\lambda, \eta) \), then \( \beta > \lambda \) so \( U_\eta \) has already been defined. Let \( V_\beta = (\phi_\lambda^\beta)^{-1}(U_\eta) \) and let \( X_\beta \) be the disjoint union of \( \text{Cl}_{X_\alpha} V \) and \( \text{Cl}_{Z_\lambda}(Z_\beta - \text{Cl}_{Z_\lambda} V_\beta) \). Let \( p_\beta \) be the projection of \( X_\beta \) onto \( Z_\beta \) and for each \( \alpha \) less than \( \beta \), let \( f_\alpha^\beta = f_\alpha^\beta \circ p_\beta \). Clearly, for \( 0 < \alpha < \gamma < \beta \), \( f_\alpha^\beta = f_\alpha^\beta \circ /_\alpha \). From [4, Lemma 2.2] it follows that \( w(X_\beta) < w(X_\gamma) \), so choose a base \( \{ U_\gamma : \gamma \in [0, \bar{\gamma}) \} \) for \( X_\beta \).

Let \( \tilde{X} \) denote the inverse limit of \( \{ (X_\alpha, f_\alpha^\beta) : \alpha < \beta < \bar{\gamma} \} \). By [4, Lemma 2.2], \( w(X) = w(\tilde{X}) \).

If, for each \( \alpha < \beta \), each bonding map \( f_\alpha^\beta \) (\( \alpha > \gamma \)) is perfect, irreducible, semi-open, onto, and continuous, then [4, Lemma 2.5] and Lemma 4 imply that \( Z_\beta \) is nonempty and each projection \( \phi_\alpha^\beta : Z_\beta \to X_\alpha \) is perfect, irreducible, semi-open, onto and continuous. Again, by [4, Lemma 2.5] and Lemmas 2 and 4, \( \tilde{X} \) is nonempty and each projection \( f_\beta : \tilde{X} \to X_\beta \) is perfect, irreducible, semi-open, onto, and continuous. Let \( f = f_0 \) be the map from \( \tilde{X} \) onto \( X \). As a subspace of a product of Hausdorff spaces, \( \tilde{X} \) is Hausdorff.

Let \( U \) be a neighborhood of \( x \in \tilde{X} \). It follows from the construction that \( x \in f_\beta^{-1}(V_\beta) \subseteq U \) for some \( \beta < \bar{\gamma} \). Since each projection is semi-open, onto, and continuous and \( \text{Cl}_{X_\alpha}(V_\beta) \) is open in \( X_\beta \), Lemma 3 implies that \( \text{Cl}_{\tilde{X}} f_\beta^{-1}(V_\beta) = f_\beta^{-1}(\text{Cl}_{X_\alpha} V_\beta) \). Hence, \( (f_\beta^{-1}(V_\beta) : \beta < \bar{\gamma}) \) is a basis for \( \tilde{X} \) and every member has an open closure.

Finally, if each \( X_\alpha \) (\( \alpha < \beta \)) is H-closed, then since each \( f_\alpha^\alpha \) and each \( \phi_\alpha^\beta \) is semi-open and onto, it follows from [7, Theorem 3.7] that each \( Z_\beta \) is H-closed and from Lemma 2 that each \( X_\beta \) is H-closed. Again by [7, Theorem 3.7], \( \tilde{X} \) is H-closed.

**Corollary.** Let \( g : X \to Y \) be a continuous map of an H-closed space \( X \) onto a semiregular, Hausdorff space \( Y \). Then \( w(Y) \leq w(X) \).

**Proof.** Let \( \tilde{X} \) be the space constructed in Theorem 1 and let \( f \) be the map from \( \tilde{X} \) onto \( X \). By Theorem 1, \( \tilde{X} \) has a base of cardinality \( w(X) \). Consider \( F = g \circ f : \tilde{X} \to Y \). Since \( \tilde{X} \) is H-closed, \( F^{-1}(y) \) is H-closed for every \( y \in Y \) [12, Theorem 1.4]. Let \( \{ U_\alpha \} \) be a base for \( \tilde{X} \) of cardinality \( w(X) \) and such that each member has an open closure. If \( \{ V_\alpha \} \) is the collection of all finite unions of closures of members of \( \{ U_\alpha \} \), then each \( V_\alpha \), and hence each \( \tilde{X} - V_\alpha \), is clopen and so H-closed. Thus, \( F(\tilde{X} - V_\alpha) \) is closed in \( Y \) for each \( \alpha \). For each \( \alpha \) let \( W_\alpha = Y - F(\tilde{X} - V_\alpha) \) and let \( y \in W_\alpha \), an open subset of \( Y \). If \( V \) is a regularly open neighborhood of \( y \) contained in \( W_\alpha \), then \( F^{-1}(y) \subseteq F^{-1}(V) \), so that by the covering characterization of H-closed spaces, \( F^{-1}(y) \subseteq V_\alpha \subseteq \text{Cl} F^{-1}(V) \) for some \( \alpha \). It follows that \( y \in W_\alpha = \text{int} W_\alpha \subseteq \text{int} \text{Cl} V = V \subseteq W_\alpha \), and hence \( \{ W_\alpha \} \) is a base for \( Y \) of cardinality \( w(X) \). But then \( w(Y) \leq w(X) \).
Remarks. It follows from [9, Theorem 1] that the absolute of the closed unit interval is not second countable. (See [6] for the definition of absolute.) This fact, together with [8, Theorem 3.4] and the preceding corollary imply the following.

Proposition. No second countable, extremally disconnected $H$-closed space can be mapped onto the closed unit interval by an irreducible, closed continuous mapping.

Bibliography


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