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SUBSHIFTS OF FINITE TYPE IN LINKED TWIST MAPPINGS

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ABSTRACT. For each pair of nonzero integers j, k, we define a homeomorphism $f_{j,k}$ of the two-disk minus three holes. We show that there exists a compact, invariant, hyperbolic set for each $f_{j,k}$ on which $f_{j,k}$ is conjugate to a subshift of finite type. This implies that the topological entropy of $f_{j,k}$ is bounded below by $4|j|\,|k| - 2|j| - 2|k|$.

The object of this note is to describe some of the orbit structure of a class of homeomorphisms of the two-disk minus three holes. We call these mappings linked twist maps. They have arisen in several different settings. W. Thurston has used similar mappings in his study of diffeomorphisms of surfaces [9]. R. Easton has conjectured that linked twist maps are ergodic [4]. And a type of linked twist map occurs in the Störmer problem, the classical mechanical system describing the motion of a charged particle in the field of a magnetic dipole [3].

I. Linked twist mappings. Let A be an annulus in the plane centered at the origin and having outer radius 3 and inner radius ρ where $2 \leqslant \rho < 3$. The twist map T of A is defined in polar coordinates by

$$T(r, \theta) = \left(r, \theta + 2\pi\left(\frac{r-\rho}{3-\rho}\right)\right). \tag{$*$}$$

T fixes the boundaries of A and rotates each interior circle $r =$ constant by an angle of $2\pi(r - \rho)/(3 - \rho)$. Also, T is an area-preserving diffeomorphism of A.

Now let A_i, $i = 1, 2$, be two copies of A : A_1 centered at $(1, 0)$, and A_2 centered at $(-1, 0)$. The union $A_1 \cup A_2$ is a space M which is homeomorphic to a closed two-disk minus three holes.

Let T_i denote the twist map on A_i. Since T_i fixes the boundary of A_i, we may extend each T_i to all of M by defining $T_i =$ identity on $M - A_i$. The resulting maps are homeomorphisms of M, which we again denote by T_i. For each pair of integers j, k with $j, k \neq 0$, we define a homeomorphism $f_{j,k} = T_2^j \circ T_1^k$. We call $f_{j,k}$ a *linked twist map*.

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A natural conjecture would be that $f_{j,k}$ is ergodic. R. Bowen has asked whether or not $f_{j,k}$ possesses homoclinic points **[2]**. We answer this question in the affirmative below. Our main result concerning linked twist maps is

THEOREM A. *There is a compact, invariant, hyperbolic set $\Lambda_{j,k} \subset M$ on which $f_{j,k}$ is topologically conjugate to the subshift of finite type generated by the matrix*

$$\mathcal{A} = \begin{pmatrix} \alpha & \alpha - 1 \\ \alpha - 1 & \alpha \end{pmatrix}$$

where $\alpha = 2|j|\,|k| - |j| - |k| + 1$.

For the definitions and basic properties of subshifts of finite type, we refer to **[5]**, **[10]**. The invariant set $\Lambda_{j,k}$ has measure zero in M, so this result does not settle the question of ergodicity of $f_{j,k}$. Also, $\Lambda_{j,k}$ is not a basic set in the usual sense, as each point in M is nonwandering. However, the proof below identifies $\Lambda_{j,k}$ as a type of local nonwandering set.

The topological entropy is a measure of how complicated the orbit structure of a mapping is. One may use a theorem of Manning to show that the entropy $h(f_{j,k})$ of linked twist maps is positive **[6]**. Theorem A gives another proof of this fact.

COROLLARY B. *The topological entropy of linked twist maps satisfies*

$$h(f_{j,k}) \geqslant \log(4|k|\,|j| - 2|j| - 2|k|).$$

In particular, $h(f_{j,k}) > 0$ *if* $|jk| \geqslant 2$.

PROOF. It is known **[1]** that $h(f) \geqslant h(f|I)$ for any homeomorphism f and compact f-invariant set I. For $\Lambda_{j,k}$, we have

$$h(f|\Lambda_{j,k}) = h(\sigma_{\mathcal{A}})$$

where $\sigma_{\mathcal{A}}$ is the shift map given by Theorem A. By the results of **[2]**,

$$h(\sigma_{\mathcal{A}}) = \log \lambda$$

where λ is the largest eigenvalue of the matrix $\mathcal{A}$. One calculates immediately that $\lambda = 4|k|\,|j| = 2|j| - 2|k|$. Q.E.D.

A second corollary of Theorem A answers the question of Bowen mentioned above. The proof is a direct application of "canonical coordinates" **[1]** and hence is omitted.

COROLLARY C. *There exist infinitely many transverse homoclinic points for $f_{j,k}$ in $\Lambda_{j,k}$.*

II. Existence of the subshift. We consider in this section the linked twist map given by $f_{j,k} = T_2^j \circ T_1^k$ where both $j, k > 0$. Minor modifications below are necessary in case either k or j is negative.

Let Q denote the component of $A_1 \cap A_2$ contained in the upper half plane, and let Q' be the other component. Let A, B, C, D denote the vertices of Q, where we assume that T_1^k fixes the sides AD and BC. See Figure 1 for the exact ordering. We wish to define a smaller quadrilateral contained in Q. Toward that end we first define four smooth curves in Q. Let $\gamma_1 \subset Q$ be a smooth curve beginning at D, terminating at some point in AB, and satisfying $T_1^k(\gamma_1) \subset CD$. γ_1 is uniquely described by these conditions. Similarly, let γ_2 be the curve in Q beginning at B, terminating in CD, and satisfying $T_1^k(\gamma_2) \subset AB$. Let σ_1 begin at D, terminate in BC, and satisfy $T_2^{-j}(\sigma_1) \subset AD$. And finally, let σ_2 begin at B, terminate in AD, and satisfy $T_2^{-j}(\sigma_2) \subset BC$.

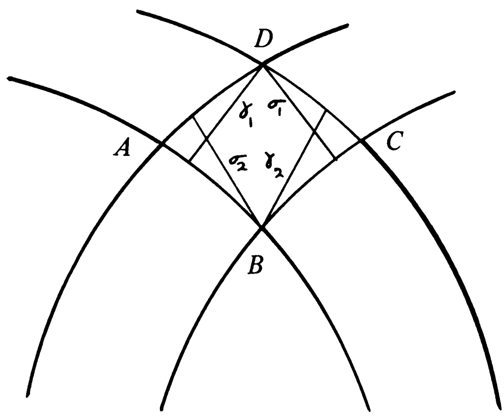

FIGURE 1

By removing the ends of each of these curves, one may check easily that these four curves describe a closed curve which bounds a "quadrilateral" R in Q. Let the vertices of R be given by a, b, c, d where $b = B$, $d = D$, and $a = \gamma_1 \cap \sigma_2$, $c = \sigma_1 \cap \gamma_2$. Using the same process, one may construct a second quadrilateral $R' \subset Q'$. Alternatively, one may simply rotate A_1 so that R "fits inside" Q'. Let a', b', c', d' denote the corresponding vertices of R'.

Finally, by choosing ρ close enough to 3, we may guarantee that $T_1^{-k}(ad) \cap Q' = \varnothing$, $T_1^{-k}(bc) \cap Q' = \varnothing$, and also that $T_2^j(cd) \cap Q' = \varnothing$, $T_2^j(ab) \cap Q' = \varnothing$.

The invariant set $\Lambda_{j,k}$ is then given by $\bigcap_{n=-\infty}^{\infty} f_{j,k}^n(R \cup R')$. That is, $\Lambda_{j,k}$ consists of all points whose forward and backward orbit remain for all time in $R \cup R'$. Clearly, $\Lambda_{j,k}$ is compact and $f_{j,k}$-invariant. We prove below that $f_{j,k}|\Lambda_{j,k}$ is topologically conjugate to a subshift of finite type. In the next section we show that $\Lambda_{j,k}$ is a hyperbolic set.

Under the map T_1^k, the sides ad and bc of R are contracted and mapped into the boundary of A_2, while the sides ab and cd are expanded and wrapped around A_1 almost k full circuits. However, $T_1^k(ab)$ and $T_1^k(cd)$ cut across R only $k - 1$ times. Hence, $T_1^k(R) \cap R$ consists of $k - 1$ strips, each connecting ab to cd. On the other hand, $T_1^k(R) \cap R'$ consists of k strips, each connecting $a'b'$ to $c'd'$. Similarly, $T_1^k(R') \cap R$ consists of k strips each of which is parallel to the strips in $T_1^k(R) \cap R$. Also, $T_1^k(R') \cap R'$ consists of $k - 1$ strips parallel to the strips in $T_1^k(R) \cap R'$.

Now apply T_2^j to a strip S_1 in either $T_1^k(R) \cap R$ or $T_1^k(R') \cap R$. T_2^j wraps

S_1 around A_2 almost j times; however, $T_2^j(S_1) \cap R$ consists of exactly $j - 1$ strips each connecting ad to bc. Also, $T_2^j(S_1) \cap R'$ consists of j strips, each connecting $b'c'$ to $a'd'$. For a strip S_2 in either $T_1^k(R) \cap R'$ or $T_1^k(R') \cap R'$, one observes similarly that $T_2^j(S_2) \cap R$ consists of j strips while $T_2^j(S_2) \cap R'$ consists of $j - 1$ strips, each parallel to the images of $T_2^j(S_1)$.

Consequently, $f_{j,k}(R) \cap R$ consists of α strips, where $\alpha = 2|j|\,|k| - |j| - |k| + 1$, and $f_{j,k}(R) \cap R'$ consists of $\alpha - 1$ strips. Also, $f_{j,k}(R') \cap R$ consists of $\alpha - 1$ strips while $f_{j,k}(R') \cap R'$ consists of α strips.

Since $f_{j,k}^{-1}$ is also a linked twist map, similar considerations apply to this map also. One may check easily that $f_{j,k}^{-1}(R) \cap R$ consists of α strips, this time connecting ab to cd, while $f_{j,k}^{-1}(R) \cap R'$ consists of $\alpha - 1$ similar strips. Also, $f_{j,k}^{-1}(R') \cap R$ consists of $\alpha - 1$ parallel strips, while $f_{j,k}^{-1}(R') \cap R'$ consists of α strips. We leave the details to the reader.

Now $f_{j,k}$ maps each strip in $f_{j,k}^{-1}(R \cup R') \cap (R \cup R')$ diffeomorphically onto a strip in $(R \cup R') \cap f_{j,k}(R \cup R')$. In the next section, we shall show that both $df_{j,k}$ and $df_{j,k}^{-1}$ preserve bundles of sectors over these sets. By the results of [7], it follows that $f_{j,k}$ is conjugate to the subshift of finite type generated by $\mathcal{Q}$. The conjugacy may be constructed as in [7].

III. Proof of hyperbolicity. In this section, we suppress the subscripts on both $f_{j,k}$ and $\Lambda_{j,k}$. Choose coordinates (θ_1, θ_2) on $Q \cup Q'$, where θ_i is the polar angle in A_i. Let X_i denote unit tangent vectors in the direction of θ_i. According to [7], to complete the proof of Theorem A, it suffices to show that both df and df^{-1} contract a bundle of sectors. More precisely, for each $x \in \Lambda$, define

$$S_x^+ = \{(\xi_0, \eta_0) \in T_xM \mid |\xi_0| \leqslant |\eta_0|\},$$

$$S_x^- = \{(\xi_1, \eta_1) \in T_xM \mid |\eta_1| \leqslant |\xi_1|\}.$$

Also define the sector bundles

$$S^+ = \bigcup_{x \in \Lambda} S_x^+, \qquad S^- = \bigcup_{x \in \Lambda} S_x^-.$$

We must show:

1. $df(S^+) \subset S^+$, $df^{-1}(S^-) \subset S^-$.

2. There exists $\mu \in (0, 1)$ such that if $(\xi_0, \eta_0) \in S_x^+$ and $df(\xi_0, \eta_0) = (\xi_1, \eta_1)$, then

a. $|\eta_1| > \mu^{-1}|\eta_0|$ if $(\xi_0, \eta_0) \in S^+$,

b. $|\xi_1| < \mu|\xi_0|$ if $(\xi_1, \eta_1) \in S^-$.

We prove 1 and 2a for f; the proof of 1 and 2b for f^{-1} is similar.

Observe first that for $x \in Q \cup Q'$, $X_2(x) = A(x)(\partial/\partial r_1)(x) + B(x)\,X_1(x)$ where $0 < \alpha_1 \leqslant |A(x)| \leqslant \alpha_2$ and $|B(x)| \leqslant \alpha_3$, provided ρ is close enough to 3, where the $\alpha_j, j = 1, 2, 3$, are constants. Hence

$$dT_1^k(X_2) = C(x)X_2 + k\lambda D(x)X_1$$

where $0 < \beta_1 \leqslant |C(x)|$, $|D(x)| \leqslant \beta_2$, and where the "shear constant" $\lambda > 0$ depends only on the inner radius ρ. From $(*)$, it follows that $\lambda \to \infty$ as

$\rho \to 3$. One has a similar formula for $dT_2^j(X_1)$:

$$dT_2^j(X_1(y)) = E(y)X_1 + j\lambda F(y)X_2$$

where again $0 < \beta_3 \leqslant |E(y)|, |F(y)| \leqslant \beta_4$. Using the fact that $dT_i(X_i) = X_i$, one then computes easily that

$$df_x = \begin{pmatrix} \phi_{11} & \phi_{12}k\lambda \\ \phi_{21}j\lambda & \phi_{22}jk\lambda^2 \end{pmatrix}$$

where the ϕ_{ij} are smooth in $Q \cup Q'$ and satisfy $0 < c_1 \leqslant |\phi_{ij}| \leqslant c_2$ for some constants c_1 and c_2.

Now suppose $(\xi_0, \eta_0) \in S_x^+$ and $df_x(\xi_0, \eta_0) = (\xi_1, \eta_1)$. We have

$$|\xi_1| \leqslant |\phi_{11}|\ |\xi_0| + |\phi_{12}|k\lambda|\eta_0| \leqslant (1 + k\lambda)|\eta_0|$$

and also

$$|\eta_1| \geqslant c_1kj\lambda^2|\eta_0| - c_2j\lambda|\eta_0| = j\lambda|\eta_0|(c_1k\lambda - c_2). \qquad (**)$$

Now, for ρ sufficiently close to 3, (**) shows that 2 above holds. Also, since $\lambda \to \infty$ as $\rho \to 3$, we have

$$1 + k\lambda \leqslant j\lambda(c_1k\lambda - c_2)$$

which implies that $|\xi_1| < |\eta_1|$. Hence $df(S_x^+) \subset S_{f(x)}^+$. This completes the proof of Theorem A.

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