CERTAIN IDEMPOTENTS LYING IN THE CENTRALIZER OF THE GROUP OF UNITS

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Abstract. Let \( S \) be a compact connected monoid of dimension \( n \) having \( G \) as a connected group of units. Let \( B \) be a closed subgroup outside of the minimal ideal. The maximum dimension possible for the product \( BG \) is \( n - 1 \). If this maximum is attained by \( BG \) and \( GB \) and both are Lie groups then \( B \) meets the centralizer of \( G \).

Let \( S \) be a compact connected monoid whose group of units \( G \) is connected. Our purpose here is to show that certain subgroups of \( S \) must meet, and in some cases lie in \( Z(G, S) \), the centralizer of the group of units. These results stem in part from the manner in which an (algebraically) irreducible submonoid is embedded. For example, it is shown [1] that an irreducible submonoid cannot have dimension at the unit exceeding \( \dim S - \dim G - 1 \). There, the important item was the structure of the product \( GB \) where \( B \) was a subgroup of \( S \).

If the closed subgroup \( B \) lies outside of the minimal ideal the maximum dimension possible for \( GB \) is \( \dim S - 1 \). It is this case which we shall pursue. If \( S \) is finite dimensional with \( G \) and \( B \) both Lie group then the condition

\[
\dim BG = \dim GB = \dim S - 1
\]

will cause \( B \) to meet \( Z(G, S) \). If \( G \) happens to be semisimple, \( \dim S - 2 \) will suffice in place of \( \dim S - 1 \).

Here, we shall be concerned with the products of the form \( GBG \). We show that if \( B \) is normal in its maximal subgroup then \( GBG \) is the space of a fiber bundle where the fiber is the homogeneous \( BG \) and the base is a quotient of \( G \).

Suppose now that \( S \) and \( G \) are as above and that \( e \) is an idempotent belonging to the compact subgroup \( B \). Further, suppose that \( B \) is disjoint from the minimal ideal as well as \( G \).

In order to show that a given \( e \) is in \( Z(G, S) \), we must show that \( Ge \subseteq eS \) and \( eG \subseteq Se \). This is equivalent to \( Ge \cup eG \subseteq H_e \)–the maximal subgroup of \( e \). Another equivalent formulation is that \( Ge \subseteq BG \) and \( eG \subseteq GB \). We first concentrate upon the condition that \( Ge \subseteq BG \). The following definition is appropriate:

Definition. Let \( Y = Y(B, G) = \{ g | g \in G, gBG = BG \} \). Clearly, \( Y \) is a...
closed subgroup of $G$. The elements of $Y$ may be described in a number of ways: (1) $g \in Y$; (2) $gBG \subseteq BG$; (3) $gB \subseteq BG$; (4) $ge \in \cap \{BGb: b \in B\}$.
Consider, however, the following condition
\[ gBG \cap BG \neq \emptyset. \]
There is no apparent reason to suppose that this condition will place $g$ in $Y$.
A moment's reflection, however, indicates that this condition is what is needed so that the sets $\{gBG\}$, $g \in G$, do form a decomposition.

Definition. We shall say that $B$ is meshed with $G$, (on the left), if $gBG \cap BG \neq \emptyset$ implies $g \in Y$, i.e. $gBG = BG$.
The appropriateness of this notion is shown in the following proposition which is fundamental.

**Proposition 1.** Let $G$ be the group of units of a compact monoid and let $B$ be a compact subgroup outside of $G$ and the minimal ideal. If $B$ is meshed with $G$ then there is an open surjection $p: GBG \to G/ Y$ given by $p(gBG) = gY$ with $p^{-1}(gY) = gBG$. Thus, $(GBG, p, G/ Y)$ is a fiber space. If $\dim G/ Y$ is finite then $q: G \to G/ Y$ admits a local cross-section and the fiber space $p: GBG \to G/ Y$ is a locally trivial fiber bundle with fiber $BG$. The associated bundle is $G \to G/ Y$.

**Proof.** Since $B$ is meshed with $G$ it is clear that $p$ is well defined and that $p^{-1}(gY) = gBG$. The fibers $[gGB]$ are clearly all homeomorphic with $BG$. Let $0$ be an open set in $GBG$ and let $V$ and $W$ be such that $V \times W$ is the inverse image of $0$ under the multiplication $G \times BG \to GBG$. Now $p(VBG) = VY$ which is certainly open in $G/ Y$ since $G \to G/ Y$ is open. But $p(0) = p(VBG)$. Thus $p$ is open. Suppose now that $G/ Y$ is of finite dimension. We know from [5] that local cross-sections exist. Thus let $U$ be an open set in $G/ Y$ and $s: U \to G$ a continuous map with $qs(u) = u$ for $u \in U$. Define the homeomorphisms $h$ and $f$ by
\[
\begin{align*}
h: U \times Y &\to q^{-1}(U), & h(u, y) &= s(u) \cdot y, \\
f: U \times BG &\to p^{-1}(U), & f(u, x) &= s(u) \cdot x.
\end{align*}
\]
Then the following diagrams commute. The first indicates the bundle structure of $q: G \to G/ Y$, which is well known and the second that of $p: GBG \to G/ Y$.

The unmarked arrows being projections and $p$ and $q$ appropriately cut down. Now let $w \in U \cap U_0$ with $U_0$ and $s_0$ playing the role of $U$ and $s$. Combining the transition functions
\[ f_0^{-1}f(w, x) = (w, s_0^{-1}(w)s(w)x). \]
Hence the associated transformations of the fiber are of the form \( x \rightarrow x_0^{-1}(w)s(w)x \) with \( s_0^{-1}(w)s(w) \in Y \). These are the transformations of the principal bundle \( g: G \rightarrow G/Y \). By definition, the associated bundle has \( G/Y \) as base and \( Y \) as fiber. We see that \( q: G \rightarrow G/Y \) is indeed the associated bundle. The condition that \( B \) be meshed with \( G \) is not unreasonable. We see this now.

**Lemma 1.** If \( eGe \cap H_e \) lies in the normalizer of \( B \) in \( H_e \) then \( B \) is meshed with \( G \). In particular, if \( C \) is the component of \( e \) in \( H_e \) then \( C \) and \( G \) are meshed.

**Proof.** Suppose \( gBG \cap BG \neq \emptyset \). Let \( x_1, x_2 \in B, y_1, y_2 \in G \) with \( gx_1y_1 = x_2y_2 \). Clearly \( gx_1G = x_2G \) and \( Bgx_1G = BG \). We may then write

\[
Bgx_1G = Bgex_1G = egeBx_1G = egBG = BG.
\]

But \( gBG \) lies entirely in the \( R \)-class of \( e \) since \( gBG \) meets \( BG \). But then \( egBG = gBG \) so that \( gBG = BG \).

When studying the fiber space \( GBG \rightarrow G/Y \) we are often concerned with the conclusion that \( G/Y \) be nondegenerate. In the important case in which \( B \) is a group component we have the following:

**Lemma 2.** Let \( S \) be a compact monoid whose group of units \( G \) is connected. Let \( C \) be the component of \( e \) in \( H_e \), where \( e^2 = e \). If \( GC \neq C \) then \( G/Y \neq \emptyset \).

**Proof.** First, \( GC \) lies in the \( \mathbb{R} \)-class of \( e \). We cannot have \( GC \subseteq H_e \) since this would imply \( GC = C \). Thus, part of \( GC \) lies outside of the \( \mathbb{R} \)-class of \( e \). But the last contains \( CG \). Thus \( GC \neq CG \) so that \( GCG \neq CG \). Then, by definition, \( G/Y \neq 0 \) where \( Y = \{ g \in GC : GC \subseteq CG \} \).

Let us note the following fact: If \( G \) is a group of units and \( B \) is a subgroup then \( BG = B \) implies \( GB \) is a left simple semigroup and hence a left group. In effect, \( GBg = GBb = GB \).

In order to continue our study of \( GBG \) we recall some material from [1].

**Definition.** Let \( W = \{ g \in G : gB = B, g \in B \} \).

Observe that \( g \in W \) if and only if \( gB \subseteq B \) if and only if \( ge \in B \).

**Lemma 3.** The group \( G \times B \) acts on \( Se \) via \( (b, g) \cdot x = gxg^{-1} \). Moreover, the map \( g \rightarrow (g, ge) \) takes \( W \) isomorphically onto the isotropy group \( (G \times B)_e \) at \( e \).

**Proof.** Observe that \( w \rightarrow we \) is a homomorphism. Furthermore, \( (g, b) \in (G \times B)_e \) if and only if \( b = ge \).

**Lemma 4.** Multiplication, \( (g, b) \rightarrow g \cdot b \) is a principal fibration \( G \times B \rightarrow GB \), with fiber \( W \). This fibration is the quotient map of a homogeneous space.

**Proof.** Since \( GB \) is the orbit of \( G \times B \) at \( e \) the map is equivalent to the quotient map defined by \( (G \times B)_e \).

For our purposes it is necessary to determine \( \dim GBG \). This is now given:

**Proposition 2.** Let \( B \) and \( G \), given as above, be meshed. If \( G \) and \( B \) are finite dimensional then \( \dim GBG = \dim BG + \dim G/Y \).

**Proof.** From the preceding, \( BG \) is locally the product of a zero
dimensional compact set and a cell of dimension equal to \( \dim BG \). Likewise \( G/Y \) has its dimensions determined by a cell of the same dimension as \( G/Y \). Since \( GBG \) is locally a product of open sets from \( BG \) and \( G/Y \), we have the desired conclusion.

Let us note that if \( G \) is connected and \( \dim G/Y = 0 \), which is the same thing as \( \dim BG = \dim GBG \), then \( Ge \subseteq BG \), by the definition of \( Y \). Thus, the larger the dimension of \( BG \) the more likely our desired conclusion.

Now let \( S \) be a compact connected finite dimensional monoid having a group of units \( G \) and a compact subgroup \( B \) outside of the minimal ideal. Then the maximum dimension possible for \( BG \) is \( \dim S - 1 \). One can see this in two different ways. Using the action of \( B \times G \) upon \( BS \) one can cite [2] directly. On the other hand, \( BG \) is a homogeneous space of \( B \times G \) and so has nonzero element in \( H^k(BG) \) where \( k = \dim BG \). This is ruled out by [7].

Definition. With \( B \) and \( G \) as above, not necessarily meshed however, we shall say that \( BG \) is of maximal dimension if \( \dim BG = n - 1 \).

**Lemma 5.** If \( BG \) is of maximal dimension then \( B \) is meshed with \( G \).

Proof. If \( C \) denotes the identity component of \( e \) in the group \( H_e \) then clearly \( \dim CG = \dim BG \). However, from the remarks before, \( C \times G \) yields the homogeneous space \( CG \) and \( B \times G \) yields the homogeneous space \( BG \). But a compact connected homogeneous space cannot contain properly another compact homogeneous space of the same finite dimension.

Suppose now that \( S, G, B \) are as above with \( \dim S = n \) and \( BG \) of maximal dimension. Then we either have \( \dim GBG = n - 1 = \dim BG \) or we have the (unlikely) situation \( \dim GBG = n \). In view of a number of considerations to follow, an appropriate way of eliminating the second possibility is that \( GBG \) have nonzero cohomology in its top dimension, i.e. \( H^{\dim GBG}(GBG) \neq 0 \). This is due to a well-known result of Wallace. See [7] or [4].

Thus, we may state the following.

**Lemma 6.** Let \( S \) be a compact connected monoid of finite dimension having a connected group of units \( G \). Let \( B \) be a compact connected subgroup (outside of the minimal ideal) such that \( GB \) and \( BG \) are of dimension \( \dim S - 1 \). Then if \( H^k(GBG) \neq 0 \), where \( k = \dim GBG \), the subgroup \( B \) meets \( Z(G, S) \).

Observe now that if \( B \) is meshed with \( G \) and both are Lie groups then \( GBG \) is a manifold. In effect \( GBG \) is fibered over the manifold \( G/Y \) by the manifold \( BG \). Thus, in this case, \( H^{\dim GBG}(GBG) \neq 0 \).

**Corollary 1.** Let \( S \) be a compact connected monoid of finite dimension whose group of units \( G \), is a connected Lie group. Suppose \( e \) is an idempotent outside of the minimal ideal and that \( C \) the group component of \( e \) is also a Lie group. If \( \dim GB = \dim BG = \dim S - 1 \) then \( e \) belongs to \( Z(G, S) \).

If \( G \) is semisimple then

\[
\dim BG \geq \dim S - 2 < \dim GB
\]

imply that \( e \) belongs to \( Z(G, B) \).
Proof. For the second claim, recall that a compact connected semisimple Lie group cannot yield a homogeneous space of dimension one. Thus, \( \dim G/Y \) is at least two unless \( GC = C \). We would then have the manifold \( GCG \) of dimension
\[
\dim G/Y + \dim CG > 2 + (\dim S - 2) = \dim S,
\]
which is impossible.

Corollary 2. Let \( S \) be a compact connected monoid of dimension \( n \), having a connected group of units \( G \). If \( e \) is an idempotent outside of \( G \) and the minimal ideal such that \( \dim H_e = n - 1 \) then \( e \in Z(G, S) \).

Proof. We note that \( GC \) is a homogeneous space of \( G \times C \). Were we to have \( C \) properly contained in \( GC \) we must have \( \dim GC = n \), since a homogeneous space which is compact and connected may not contain another of the same finite dimension. Thus \( GC = C \) and, in the same way, \( CG = C \).

Corollary 3. Let \( S \) be a compact connected finite dimensional monoid whose group of units \( G \) is a connected semisimple Lie group. Let \( e \) be an idempotent belonging to neither \( Z(G, S) \) nor the minimal ideal. If \( C \) the maximal compact connected subgroup at \( e \) is a Lie group then \( \dim C < \dim S - 3 \).

Proof. Since \( e \notin Z(G, S) \) we cannot have \( GC = CG = C \). Say \( GC \neq C \). Then consider the fibering \( GCG \to G/Y \). Since \( GC \subset C \) we know that \( GC \subset CG \). Thus the quotient \( G/Y \) is nondegenerate. Since \( G \) is semisimple \( \dim G/Y > 2 \). Now, \( \dim GCG = \dim CG + \dim G/Y \). Thus, if \( \dim C = \dim S - 2 \) we would have \( \dim GCG = \dim S \) which, as we know, is impossible. Hence \( \dim C < \dim S - 3 \).

Corollary 4. Let \( S \) be a compact connected monoid having a group of units \( G \) which is a connected Lie group. Let \( e \) be an idempotent belonging to neither \( Z(G, S) \) nor the minimal ideal. Let \( C \) denote the component of \( e \) in \( H_e \) and suppose that \( C \) is a Lie group. If either

1. \( \dim C = \dim S - 2 \) or
2. \( G \) is semisimple and \( \dim C = \dim S - 3 \),

holds then either \( GC \) or \( CG \) (not both) coincides with \( C \). Thus, say, \( GC \) is a left group with \( C \) as maximal subgroup.

Proof. Consider again \( GCG \to G/Y \). Suppose first, \( \dim C = \dim S - 2 \). Then if both \( GC \neq C \) and \( CG \neq C \) we would have \( \dim GCG = \dim CG + \dim G/Y \). Since \( C \subset CG \) we have \( \dim CG > \dim S - 1 \). Since \( GC \neq C \) we have \( GCG \subset CG \) so that \( \dim G/Y > 1 \). Thus, \( \dim GCG = \dim S \) which is impossible. Now suppose that \( \dim C = \dim S - 3 \) with \( G \) semisimple. In this case if \( GC \neq C \) and \( CG \neq C \) we have
\[
\dim GCG = \dim CG + \dim G/Y > \dim S - 2 + 2 = \dim S.
\]
Again, a contradiction.

Example. Let \( G \) be the three sphere as a topological group using
quaternions of norm one. Following [3] let \(|G|\) be the space \(G\) with left zero multiplication. Then \(L = \{|G| \times G\}\) is a left group. Let \(G\) act on \(L\) by 
\[g(g', g'') = (gg', gg'')\] and 
\[(g', g'')g = (g', g''g)\]. Then \(L \cup G\) is a compact monoid in which \(e = (1, 1)\), where \(1\) is the unit of \(G\), is not in the centralizer of \(G\). Instead of \(G\) one may use some homogeneous space of \(G\) say \(G/K\). Again \(G \cup \{|G/K\}G\) is a monoid. We may take \(S = G \cup \{|G/K\}G\) of dimension five with \(G/K\) say \(S^2\). Now take the cone over \(S\) and get a compact connected monoid of dimension six with \(e\) having its component of dimension three = \(n - 3\). Here \(e\) is not in the centralizer of the group of units. However the set corresponding to \(GC\) is again a left group. See [3] for further details.

As the reader has noted, the fibering of \(GBG\) by \(BG\) over \(G/Y\) shows that the total space has nontrivial cohomology in top dimension if \(B\) and \(G\) are manifolds, i.e. Lie groups. It thus seems reasonable to conjecture that if \((E, B, F)\) is a fibre bundle with space \(E\), base \(B\) and fibre \(F\) then \(B\) and \(F\) are compact connected finite dimensional topological groups one could conclude that \(H^n(E, Z) \neq 0\) where \(n = \dim B + \dim F\). (It is known that \(H^{\dim B}(B)\) and \(H^{\dim F}(F)\) are nonzero over \(Z\).) Oddly enough such is not the case.

**Example.** There exists a space \(E\) which is a fibre bundle over the circle fibered by the dyadic solenoid with \(H^2(E, Z) = 0\).

Let \(K\) be the Klein bottle obtained as usual by identification of the ends of a cylinder \(I \times T\) through the diameter. On \(T\) the circle this is the map \(Z \rightarrow Z^{-1}\). Since this identification map is compatible with the squaring map \(Z \rightarrow Z^2\) on \(T\) we may map \(K\) onto itself preserving the base and wrapping each fibre twice. This map \(\alpha\) induces \(\alpha^*\) on \(H^2(K, Z)\) which is trivial.

Recall that \(H^2(K, Z) = Z_2\). From direct considerations, the kernel of \(\alpha^*\) is again \(Z_2\), i.e. all of \(H^2(K, Z)\). The inverse limit of a square of such spaces \(K\) and maps \(\alpha\) will yield a fibre bundle \(E\) with base a circle and fibre the dyadic solenoid. However,

\[H^2(E, Z) = 0\]

We mention that Floyd has constructed a free action of \(p\)-adic group on an acyclic space [6].

**References**


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