

A NOTE ON STABLE EQUIVALENCE AND NAKAYAMA ALGEBRAS

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ABSTRACT. Two artin algebras Λ and Λ' are said to be stably equivalent if the categories of finitely generated modules modulo projective for Λ and Λ' are equivalent categories.

If Λ' is stably equivalent to a Nakayama (i.e. generalized uniserial) algebra Λ , we prove that Λ and Λ' have the same number of nonprojective simple modules. And if Λ and Λ' are stably equivalent indecomposable Nakayama algebras where each indecomposable projective module has length at least 3, then Λ and Λ' have the same admissible sequences.

Introduction. We recall that two artin algebras Λ and Λ' are said to be stably equivalent if the categories $\underline{\text{mod}} \Lambda$ and $\underline{\text{mod}} \Lambda'$ modulo projectives are equivalent categories, where $\text{mod } \Lambda$ denotes the category of finitely generated left Λ -modules.

An open problem about stable equivalence, which is of importance in the modular representation theory of finite groups, is whether two stably equivalent algebras Λ and Λ' have the same number of nonprojective simple modules. In §1 we show that if one of the algebras is a Nakayama (i.e. generalized uniserial) algebra, then this is the case. It is also known to be true if Λ is hereditary [2, Theorem 5.5] and if Λ has no nonzero projective injective modules [1, Proposition 3.6]. See also [11].

In §2 we prove the result, announced in [10], that if Λ and Λ' are stably equivalent Nakayama algebras such that each indecomposable projective module has length at least 3, then Λ and Λ' have the same admissible sequences (see §2 for definition). Hence if they are both basic algebras over an algebraically closed field, they must be isomorphic [8, Satz 7]. Our method is based upon using the notion of irreducible map introduced in [3].

In connection with the above, we point out that if Λ' is an artin algebra stably equivalent to a Nakayama algebra Λ , then Λ' is not necessarily Nakayama [9]. If, however, Λ has Loewy length at most 3, then Λ' is Nakayama [9, Corollary 2.2]. However, there are non Nakayama algebras Λ' stably equivalent to Nakayama algebras of Loewy length 4, and in [12] we give, as announced in [10], a characterization of the artin algebras stably equivalent to Nakayama algebras of Loewy length 4. P. Gabriel and C.

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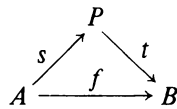
Riedtmann have studied artin algebras stably equivalent to self-injective algebras over an algebraically closed field. Stable equivalence of Nakayama algebras has also been studied in [6].

1. Let Λ be an artin algebra, and denote by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules. Denote by $\underline{\text{mod}} \Lambda$ the category $\text{mod } \Lambda$ modulo projectives, that is, the objects of $\underline{\text{mod}} \Lambda$ are the same as those of $\text{mod } \Lambda$, now denoted \underline{M} for M in $\text{mod } \Lambda$, and $\underline{\text{Hom}}(\underline{M}, \underline{N}) = \text{Hom}_\Lambda(M, N)/P(M, N)$, where $P(M, N)$ consists of the maps from M to N which factor through a projective Λ -module. For $f: M \rightarrow N$, let \underline{f} denote the image of f in $\underline{\text{Hom}}(\underline{M}, \underline{N})$. Λ and Λ' are said to be stably equivalent if there is an equivalence of categories $\alpha: \underline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda'$. α induces a correspondence, also denoted by α , between the objects in $\text{mod}_P \Lambda$ and in $\text{mod}_P \Lambda'$, the categories of finitely generated modules with no nonzero projective summands.

The following lemmas about stable equivalence will be useful in what follows.

LEMMA 1.1. *Let Λ be an artin algebra and $f: A \rightarrow B$ an epimorphism in $\text{mod } \Lambda$. If A and B have no nonzero projective summands, then \underline{f} is not zero.*

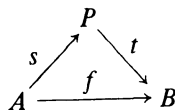
PROOF. If \underline{f} was zero, we would have some commutative diagram



where P is projective. Since A has no nonzero projective summand $\text{Im } s$ must be contained in rP , where r denotes the radical of Λ . Hence $\text{Im } f \subset rB$, which contradicts the fact that f is an epimorphism. It follows that \underline{f} is not zero.

LEMMA 1.2. *Let Λ be a Nakayama algebra and $f: A \rightarrow B$ a monomorphism. If A and B are indecomposable nonprojective Λ -modules, then \underline{f} is not zero.*

PROOF. Assume that \underline{f} is zero. We then have some commutative diagram



where $t: P \rightarrow B$ is a projective cover. Since P is indecomposable, $\text{soc } P$ is simple, and since $t: P \rightarrow B$ is not a monomorphism, $t(\text{soc } P)$ must be zero. It follows that $f(\text{soc } A) = ts(\text{soc } A)$ is zero, since $s(\text{soc } A) \subset \text{soc } P$. This contradicts the fact that f is a monomorphism. Hence it follows that \underline{f} is not zero.

In this section we prove the following result, which under the assumption that Λ and Λ' are self-injective, is essentially proved in [7].

THEOREM 1.3. *Let Λ and Λ' be stably equivalent artin algebras. If Λ is*

Nakayama, then Λ and Λ' have the same number of nonprojective simple modules.

PROOF. Let $\alpha: \text{mod } \Lambda' \rightarrow \text{mod } \Lambda$ be an equivalence, where Λ is Nakayama. Let S_1, \dots, S_n be the nonprojective simple Λ' -modules. We then claim that $\text{soc}(\alpha S_1), \dots, \text{soc}(\alpha S_n)$ are nonisomorphic simple Λ -modules. Since Λ is Nakayama, and the αS_i are all indecomposable, we know that the $\text{soc}(\alpha S_i)$ are all simple. Assume that for $i \neq j$, $\text{soc}(\alpha S_i) \cong \text{soc}(\alpha S_j)$. We can assume without loss of generality that the length of αS_i is at most equal to the length of αS_j . Since Λ is Nakayama, we know by the structure of indecomposable Λ -modules that there is a monomorphism $t: \alpha S_i \rightarrow \alpha S_j$. Since $\text{Hom}(S_i, S_j) = 0$, we have $\text{Hom}(\alpha S_i, \alpha S_j) = 0$, which is a contradiction by Lemma 1.2. This shows that the $\text{soc}(\alpha S_i)$ are all nonisomorphic.

We further observe that since $\text{soc}(\alpha S_i)$ is a composition factor of some $P_i/\text{soc } P_i$, where P_i is indecomposable projective, $\text{soc}(\alpha S_i)$ is not a simple projective Λ -module. Hence Λ has at least n nonisomorphic simple nonprojective modules.

Let now T be a simple nonprojective Λ -module, and let A be the indecomposable nonprojective Λ' -module such that $\alpha A = T$. Then there is an epimorphism $t: A \rightarrow S_i$ for some i , hence we know by Lemma 1.1 that $\text{Hom}(A, S_i)$ is not zero. It then follows that $\text{Hom}(T, \alpha S_i)$ is not zero. This means that there is a nonzero map, hence a monomorphism, from T to αS_i , so that $T \cong \text{soc } \alpha S_i$. This shows that Λ has exactly n simple nonprojective modules, and the proof is done.

2. Let Λ be a Nakayama algebra. We shall recall some well known facts about indecomposable Nakayama algebras (see [8], [5]). There is an ordering of the indecomposable projective Λ -modules P_1, \dots, P_n , such that $P_i/rP_i \cong rP_{i+1}/r^2P_{i+1}$, $i = 1, \dots, n-1$, and either rP_n is zero or $P_n/rP_n \cong rP_1/r^2P_1$. The sequence of integers c_1, \dots, c_n , where c_i is the length of P_i , is called an admissible sequence for Λ . An admissible sequence is determined up to cyclic permutation. If Λ is a basic indecomposable Nakayama algebra over an algebraically closed field k , then the associated admissible sequence uniquely determines Λ [8]. We recall that we showed in [9] that if Λ and Λ' are stably equivalent indecomposable Nakayama algebras, with no simple projective modules then the associated admissible sequences have the same length, that is, the same number of c_i (this also follows from Theorem 1.1, since it says that Λ and Λ' have the same number of nonprojective simple modules), and that the number of c_i equal to a given number $t \geq 2$ is the same for both admissible sequences. (See [9, Proposition 1.1].) In the special case that all c_i are 3 or 4, we showed that the admissible sequences are actually the same ([9, Proposition 3.1], [10, Proposition 3.1']). Here we shall, as announced in [10], prove the same thing just assuming that all c_i are at least 3. We give an example to show that the result is not true in general.

To do this we shall use the notion of irreducible maps introduced in [3]. We recall that for an artin algebra Λ , a map $f: A \rightarrow B$ in $\text{mod } \Lambda$ is said to be

irreducible if f is neither a split epimorphism nor a split monomorphism, and given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g & \nearrow h \\ & & X \end{array}$$

then either g is a split monomorphism or h is a split epimorphism. It is not hard to see that if Λ is a Nakayama algebra, then a map $f: A \rightarrow B$ between indecomposable modules A and B in $\text{mod } \Lambda$ is irreducible if and only if f is a monomorphism and $l(A) = l(B) - 1$ or f is an epimorphism and $l(A) = l(B) + 1$ (see [3]). Here l denotes length. Another useful fact about irreducible maps that we shall need here is that if $\alpha: \text{mod } \Lambda \rightarrow \text{mod } \Lambda'$ is an equivalence and $f: A \rightarrow B$ is an irreducible map where A and B are in $\text{mod}_p \Lambda$, then $f': A' \rightarrow B'$, where $A' = \alpha(A)$, $B' = \alpha(B)$, $\alpha(f) = f'$, is also irreducible [4, Proposition 12].

To prove that two stably equivalent indecomposable Nakayama algebras, where each indecomposable projective module has length at least 3, have the same admissible sequences, we shall need a few lemmas.

LEMMA 2.1. *Let Λ be a Nakayama algebra.*

(a) *Let S_1 be a simple Λ -module and assume that the projective cover P_1 of S_1 has length at least 3. Then $S_2 = \tau P_1 / \tau^2 P_1$ is the unique indecomposable nonprojective Λ -module with the property that there is some chain $S_2 \xrightarrow{f} X \xrightarrow{g} S_1$ of irreducible maps between indecomposable nonprojective Λ -modules such that gf is zero.*

(b) *Let P_1 be an indecomposable projective Λ -module of length $n \geq 3$. Then there is a unique indecomposable nonprojective Λ -module A such that there is some chain $A \xrightarrow{f} Y \xrightarrow{g} P_1 / \tau^{n-1} P_1$ of irreducible maps between indecomposable nonprojective modules such that gf is zero, if and only if τP_1 is not projective.*

If τP_1 is not projective, then $A \cong \tau P_1$.

PROOF. (a) $P_1 / \tau^2 P_1$ is the only indecomposable (nonprojective) Λ -module such that there is some irreducible map $P_1 / \tau^2 P_1 \rightarrow S_1$. The only indecomposable Λ -modules having irreducible maps to $P_1 / \tau^2 P_1$ are $P_1 / \tau^3 P_1$ and $\tau P_1 / \tau^2 P_1 = S_2$. S_2 is not projective, and for any chain

$$S_2 \xrightarrow{f} P_1 / \tau^2 P_1 \xrightarrow{g} S_1$$

of irreducible maps, gf is clearly zero. And if $P_1 / \tau^3 P_1$ is not projective, and $P_1 / \tau^3 P_1 \xrightarrow{f} P_1 / \tau^2 P_1 \xrightarrow{g} S_1$ is a chain of irreducible maps, then gf is an epimorphism, so that $gf \neq 0$ by Lemma 1.1.

(b) $\tau P_1 / \tau^{n-1} P_1$ is the only indecomposable nonprojective Λ -module with an irreducible map to $P_1 / \tau^{n-1} P_1$. The indecomposable nonprojective modules with an irreducible map to $\tau P_1 / \tau^{n-1} P_1$ are $\tau^2 P_1 / \tau^{n-1} P_1$, if $n \geq 4$, and τP_1 , if τP_1 is not projective. For $n \geq 4$, any composition of irreducible maps

$$\tau^2 P_1 / \tau^{n-1} P_1 \xrightarrow{f} \tau P_1 / \tau^{n-1} P_1 \xrightarrow{g} P_1 / \tau^{n-1} P_1$$

is a monomorphism. Hence \underline{gf} is not zero by Lemma 1.2. If τP_1 is not projective, let $\tau P_1 \xrightarrow{f} \tau P_1 / \tau^{n-1} P_1 \xrightarrow{g} P_1 / \tau^{n-1} P_1$ be a chain of irreducible maps between indecomposable nonprojective Λ -modules. It is then easy to see that $\underline{gf} = 0$. This finishes the proof of (b).

LEMMA 2.2. *Let $\alpha: \text{mod } \Lambda \rightarrow \text{mod } \Lambda'$ be a stable equivalence between Nakayama algebras, and let S_1 be a simple Λ -module such that $L(P_1) = n_1 \geq 3$, where P_1 is a projective cover for S_1 . Assume that $\alpha S_1 \cong T_1$, where T_1 is a simple Λ' -module, and denote by Q_1 a projective cover of T_1 . Then $\alpha S_2 \cong T_2$, where $S_2 = \tau P_1 / \tau^2 P_1$ and $T_2 = \tau Q_1 / \tau^2 Q_1$.*

PROOF. We have the following chains of irreducible maps between indecomposable nonprojective Λ' -modules.

$$\alpha S_2 \xrightarrow{f} \alpha(P_1 / \tau^2 P_1) \xrightarrow{g} \alpha S_1 = T_1 \quad \text{and} \quad T_2 \xrightarrow{f'} Q_1 / \tau^2 Q_1 \xrightarrow{g'} T_1.$$

To get the first chain, we first consider $S_2 \rightarrow P_1 / \tau^2 P_1 \rightarrow S_1$, and use the connection between stable equivalence and irreducible maps. Then we have $\underline{gf} = 0$, and also $\underline{g'f'} = 0$. By Lemma 2.1(a) we can then conclude that $\alpha S_2 \cong T_2$.

LEMMA 2.3. *Let $\alpha: \text{mod } \Lambda \rightarrow \text{mod } \Lambda'$ be a stable equivalence between indecomposable Nakayama algebras, and assume that each indecomposable projective Λ -module and Λ' -module has length at least 3. If for some simple Λ -module S_1 , αS_1 is not simple, then the admissible sequences for Λ and Λ' are both constant, and are hence the same.*

PROOF. Assume that αS_1 is not simple. $(, S_1)$ denotes the functor from $\text{mod } \Lambda$ to Ab given by $(, S_1)(A) = \text{Hom}(\underline{A}, \underline{S_1})$. By [9, Proposition 1.1] we know that $(, S_1)$ is uniserial and that $\alpha S_1 \cong Q_1 / \tau^{n-1} Q_1$, where Q_1 is an indecomposable projective Λ' -module of length $n \geq 3$. We further know that

$$l(P_1) - 1 = l((, S_1)) = l((, Q_1 / \tau^{n-1} Q_1)) = l(Q_1) - 1,$$

where P_1 denotes a projective cover for S_1 . The chain $S_2 \xrightarrow{f} P_1 / \tau^2 P_1 \xrightarrow{g} S_1$ of irreducible maps gives rise to a chain of irreducible maps $\alpha S_2 \xrightarrow{f'} \alpha(P_1 / \tau^2 P_1) \xrightarrow{g'} \alpha S_1$, where \underline{gf} is zero. By Lemma 2.1, τQ_1 is not projective, and $\alpha S_2 \cong \tau Q_1$. Since τQ_1 cannot be simple, we have as above that $\tau Q_1 \cong Q_2 / \tau^{m-1} Q_2$, where Q_2 is an indecomposable projective Λ' -module of length $m \geq 3$, and also that

$$l(P_2) - 1 = l((, S_2)) = l((, Q_2 / \tau^{m-1} Q_2)) = l(Q_2) - 1.$$

Here P_2 denotes a projective cover for S_2 . Since clearly $l(Q_1) = l(Q_2)$, we have by the above that $l(P_1) = l(P_2)$. The proof of the lemma is then easily finished.

We are now ready to prove our desired result.

THEOREM 2.4. *Let Λ and Λ' be stably equivalent indecomposable Nakayama*

algebras where each indecomposable projective module has length at least 3. Then Λ and Λ' have the same admissible sequences.

PROOF. Let $\alpha: \underline{\text{mod}} \Lambda \rightarrow \underline{\text{mod}} \Lambda'$ be an equivalence. Assume first that there is some simple Λ -module S such that αS is not simple. Then we are done by Lemma 2.3. If for every simple Λ -module S , αS is simple, we are done using Lemma 2.2 since $l(P) - 1 = l((, \underline{S}))$, where P is a projective cover for S [9]. This finishes the proof of the theorem.

We have only dealt with indecomposable Nakayama algebras. This is no loss of generality because if Λ and Λ' are stably equivalent Nakayama algebras where each indecomposable projective module has length at least 3, and $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ and $\Lambda' = \Lambda'_1 \times \cdots \times \Lambda'_m$ are decompositions into products of indecomposable algebras, then $m = n$, and the numbering can be made such that Λ_i and Λ'_i are stably equivalent. This is a direct consequence of the following result.

PROPOSITION 2.5. *Let Λ be a Nakayama algebra such that each indecomposable projective Λ -module has length at least 3. Then Λ is indecomposable if and only if $\underline{\text{mod}} \Lambda$ is an indecomposable category.*

PROOF. Assume that Λ is indecomposable, and let S_1, \dots, S_n be the simple Λ -modules, ordered in a way such that $S_{i+1} \cong rP_i/r^2P_i$, $i = 1, \dots, n-1$, $S_1 \cong rP_n/r^2P_n$. We have nonzero morphisms $S_{i+1} \rightarrow P_i/r^2P_i$ and $P_i/r^2P_i \rightarrow S_i$. And if X is an arbitrary indecomposable nonprojective Λ -module, there is clearly some S_i such that $\text{Hom}(X, S_i) \neq 0$. From this we conclude that $\underline{\text{mod}} \Lambda$ is indecomposable.

If $\underline{\text{mod}} \Lambda$ is indecomposable and Λ has no semisimple ring summand, then Λ is clearly indecomposable.

We end this section by pointing out that if Λ and Λ' are stably equivalent indecomposable Nakayama algebras, then we cannot in general conclude that they have the same admissible sequences. For example, if k is a field, then $k[x]/(x^2)$ and $\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ are stably equivalent Nakayama algebras with admissible sequences (2) and (2, 1) respectively.

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