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ON A THEOREM OF CAMBERN

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ABSTRACT. It is shown that Cambern's generalized Banach-Stone theorem for spaces of continuous functions on compact sets cannot be extended to the case of spaces of continuous affine functions on simplexes.

Introduction. The present paper is concerned with the question whether Cambern's theorem on isomorphisms with small bound for spaces $C(X)$ of continuous real functions on compact sets X can be generalized to spaces $A(K)$ of continuous affine functions on compact simplexes K. The theorem of Cambern [**1**], in question, is the following:

If X and Y are compact and if there is a linear isomorphism ϕ of $C(X)$ onto $C(Y)$ with $\|\phi\| \cdot \|\phi^{-1}\| < 2$ then X and Y are homeomorphic.[1]

In terms of continuous affine functions, this statement can be rephrased as follows:

If K_1 and K_2 are Bauer simplexes and if there is a linear isomorphism ϕ of $A(K_1)$ onto $A(K_2)$ with $\|\phi\| \cdot \|\phi^{-1}\| < 2$ then the sets of extreme points $\mathrm{ex}(K_1)$ and $\mathrm{ex}(K_2)$ are homeomorphic.

In this note, we give a negative answer to the question whether or not this result holds for arbitrary simplexes K_1 and K_2, by proving:

THEOREM. *For each* $\alpha \in]0, 1[$ *there exist two simplexes* K_1 *and* K_2 *such that* $\mathrm{ex}(K_1)$ *and* $\mathrm{ex}(K_2)$ *are not homeomorphic and such that there exists a linear isomorphism* ϕ *of* $A(K_1)$ *onto* $A(K_2)$ *with* $\|\phi\| \cdot \|\phi^{-1}\| < 1 + \alpha$.

As the proof of the theorem will show, here K_1 can be chosen to be a Bauer simplex. Note that $A(K_1)$ and $A(K_2)$ cannot possibly be isometric if $\mathrm{ex}(K_1)$ and $\mathrm{ex}(K_2)$ are not homeomorphic and if K_1 and K_2 are simplexes (see [**2**, p. 169]).

NOTATIONS. Let $\overline{N}$ denote the one point compactification of the discrete space $\mathbf{N}$ of positive integers. For every $r \in]0, 1[$ we define

$$A_r := \left\{ f \in C(\overline{N}) | f(\infty) = r \cdot f(1) + (1 - r) \cdot f(2) \right\}$$

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[1] Cambern [**1**] also proved that this result remains true if X and Y are locally compact and $C(X)$ (resp. $C(Y)$) denotes the space of continuous functions on X (resp. Y) which are zero at infinity.

and

$$K_r := \{m \in A_r^* \mid m \geq 0, m(1) = 1\}.$$

LEMMA 1. *For every $r \in]0, 1[$ the w^*-compact convex set K_r is a simplex.*

PROOF. We know that for every $r \in]0, 1[$ the Choquet boundary for A_r coincides with $\mathbf{N}$. Besides, we see that the only measures on $\bar{N}$ which annihilate A_r are the real multiples of

$$\varepsilon_\infty - r \cdot \varepsilon_1 - (1 - r) \cdot \varepsilon_2$$

and these are not supported by $\mathbf{N}$; that is the only annihilating boundary measure for A_r is 0, so K_r is a simplex.

LEMMA 2. *For every $r \in]0, 1[$ there exists a linear isomorphism ϕ_r of A_r onto $C(\bar{N})$ such that*

$$\|\phi_r(f)\| \leq \|f\| \leq (2/r - 1) \cdot \|\phi_r(f)\| \quad \textit{for all } f \in A_r.$$

PROOF. We define $\phi_r: A_r \to C(\bar{N})$ by

$$(\phi_r(f))(i) := \begin{cases} f(i + 1) & \text{if } i \in \mathbf{N}, \\ f(\infty) & \text{if } i = \infty. \end{cases}$$

The linear map ϕ_r is one to one since the equality $\phi_r(f) = 0$ implies: $f(i + 1) = 0$ for all $i \in \mathbf{N}$ and

$$0 = f(\infty) = r \cdot f(1) + (1 - r) \cdot f(2) = r \cdot f(1),$$

from which the equality $f = 0$ follows.

ϕ_r is surjective too, for if we choose $g \in C(\bar{N})$ and define

$$h(i) := \begin{cases} (1/r) \cdot (g(\infty) - (1 - r) \cdot g(1)) & \text{if } i = 1, \\ g(i - 1) & \text{if } \infty > i \geq 2, \\ g(\infty) & \text{if } i = \infty, \end{cases}$$

then $h \in A_r$ and $\phi_r(h) = g$.

Now, choose $f \in A_r$. We see immediately that $\|\phi_r(f)\| \leq \|f\|$. To show the inequality $\|f\| \leq (2/r - 1) \cdot \|\phi_r(f)\|$ we may suppose that $\|f\| > \|\phi_r(f)\|$, because otherwise $\|f\| \leq \|\phi_r(f)\| \leq (2/r - 1) \cdot \|\phi_r(f)\|$. Hence we obtain the following inequality

$$\begin{aligned} \|f\| &= \sup\{|f(i)| \mid i \in \mathbf{N}\} \\ &= \max\big(\sup\{|f(i)| \mid i \in \mathbf{N} \setminus \{1\}\}, |f(1)|\big) \\ &= \max\big(\|\phi_r(f)\|, |f(1)|\big) = |f(1)| \\ &= (1/r) \cdot |f(\infty) - (1 - r) \cdot f(2)| \leq (1/r) \cdot \big(\|\phi_r(f)\| + (1 - r) \cdot \|\phi_r(f)\|\big) \\ &= (2/r - 1) \cdot \|\phi_r(f)\|. \end{aligned}$$

This proves the assertion of Lemma 2.

Proof of the theorem. Choose $\alpha \in]0, 1[$ and put $r_\alpha = 2/(2 + \alpha)$. Then, according to Lemma 2, there is a linear isomorphism ϕ_{r_α} of A_{r_α} onto $C(\bar{N})$ with $\|\phi_{r_\alpha}\| \cdot \|\phi_{r_\alpha}^{-1}\| \leqslant (2/r_\alpha - 1) = (1 + \alpha)$. Therefore, by the usual identification of A_{r_α} with $A(K_{r_\alpha})$ and of $C(\bar{N})$ with $A(K)$, where K denotes the set of all probability measures on $\bar{N}$, we obtain the following result: For every $\alpha \in]0, 1[$ there exist $r_\alpha \in]0, 1[$ and a linear isomorphism ϕ_{r_α} of $A(K_{r_\alpha})$ onto $A(K)$ such that $\|\phi_{r_\alpha}\| \cdot \|\phi_{r_\alpha}^{-1}\| \leqslant 1 + \alpha$. Beside this, we know by Lemma 1 that K_{r_α} is a simplex. Since $\mathrm{ex}(K_{r_\alpha})$ (resp. $\mathrm{ex}(K)$) can be identified with **N** (resp. $\bar{N}$), $\mathrm{ex}(K_{r_\alpha})$ and $\mathrm{ex}(K)$ are not homeomorphic. This completes the proof of the theorem.

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References

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