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ALMOST EVERY QUASINILPOTENT HILBERT SPACE OPERATOR IS A UNIVERSAL QUASINILPOTENT[1]

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ABSTRACT. Let Q be a quasinilpotent operator acting on a complex separable infinite dimensional Hilbert space; then either Q^k is compact for some positive integer k, or the closure of the similarity orbit of Q contains every quasinilpotent operator. Analogous results are shown to be true for the Calkin algebra and for nonseparable Hilbert spaces. For the nonseparable case, the analogous result is true for the closed bilateral ideal $\mathcal{J}$, strictly larger than the ideal of compact operators, if and only if $\mathcal{J}$ is not the ideal associated with an $\aleph_0$-regular limit cardinal. For the ideal of compact operators, the problem remains open.

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all (bounded linear) operators acting on the complex separable infinite dimensional Hilbert space $\mathcal{H}$. A quasinilpotent operator Q is called a *universal quasinilpotent operator* (u.q.o.) if the closure $\mathcal{S}(Q)^-$ of its *similarity orbit*

$$\mathcal{S}(Q) = \{WQW^{-1}: W \text{ is invertible in } \mathcal{L}(\mathcal{H})\}$$

contains the set $\mathbf{Q}(\mathcal{H})$ of all quasinilpotent operators.

The existence of a u.q.o. was proven in [**7**] and a strong refinement of this result was obtained in [**1**, §7]. The main result of this note provides a complete characterization of the set of all universal quasinilpotent operators.

THEOREM 1. *Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space and let $\mathbf{N}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the set of all nilpotent operators and the ideal of compact operators, respectively. Then $Q \in \mathcal{L}(\mathcal{H})$ is a universal quasinilpotent operator if and only if it belongs to $\mathbf{Q}_{ae}(\mathcal{H}) = \mathbf{Q}(\mathcal{H}) \setminus [\mathbf{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]$.*

Furthermore, $\mathbf{Q}_{ae}(\mathcal{H})$ is a G_δ dense subset of $\mathbf{Q}(\mathcal{H})$.

Let π be the canonical projection of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{A}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. It is completely apparent that a quasinilpotent T belongs to $\mathbf{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})$ if and only if $\pi(T)$ is a nilpotent in $\mathcal{A}(\mathcal{H})$ [**10**],

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[**13**], and therefore every u.q.o. is an element of $\mathbf{Q}_{ae}(\mathcal{H})$ [**1**], [**8**, Proposition 1 (vii)]. On the other hand, we have: (a) $\mathbf{Q}(\mathcal{H})$ is a G_δ subset of $\mathcal{L}(\mathcal{H})$ [**5**, proof of Theorem (3.2)]; (b) $\pi[\mathbf{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]$ and $\mathbf{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})$ are F_σ subsets of $\mathcal{A}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$, respectively [**6**, Corollary 3]; (c) $\mathbf{Q}_{ae}(\mathcal{H})$ is a dense subset of $\mathbf{Q}(\mathcal{H})$ [**7**]; and therefore, (d) $\mathbf{Q}_{ae}(\mathcal{H})$ is a G_δ dense subset of $\mathbf{Q}(\mathcal{H})$, i.e., contains "almost every" quasinilpotent.

Thus, in order to complete the proof of Theorem 1 it remains to show that every element of $\mathbf{Q}_{ae}(\mathcal{H})$ is actually a u.q.o.; this will be done in §2 below, which also contains the analogous result for the Calkin algebra $\mathcal{A}(\mathcal{H})$. §3 contains the natural extensions of Theorem 1 to the algebra of operators and the closed bilateral ideals which are strictly larger than $\mathcal{K}(\mathcal{H})$, for nonseparable Hilbert spaces. The analogous result for $\mathcal{K}(\mathcal{H})$ (Conjecture 1 of [**7**]) remains an open problem.

2. Characterization of the set of u.q.o. Let $Q \in \mathbf{Q}_{ae}(\mathcal{H})$ and let ρ be a faithful representation of the C^*-algebra $[C^*(Q) + \mathcal{K}(\mathcal{H})]/\mathcal{K}(\mathcal{H})$, in the separable Hilbert space $\mathcal{H}_\rho$, where $C^*(Q)$ denotes the C^*-algebra generated by Q. Clearly, $Q_1 = \rho \circ \pi(Q) \in \mathbf{Q}(\mathcal{H}_\rho) \setminus \mathbf{N}(\mathcal{H}_\rho)$ and, by Voiculescu's theorem [**14**, Theorem 1.3], the closure $\mathcal{U}(Q)^-$ of the unitary orbit of Q contains an operator $Q_2 \approx Q \oplus Q_1^{(\infty)}$, where $\approx$, $\oplus$ and $T^{(\alpha)}$ denote "unitarily equivalent to", "*orthogonal* direct sum" and "orthogonal direct sum of α $(0 \leqslant \alpha \leqslant \infty)$ copies of T", respectively.

Since Q and Q_1 are quasinilpotent operators, they are quasitriangular [**2**] and therefore there exist compact operators $K_0, K_1, K_2, \ldots,$ such that $\|K_n\| < 2^{-n}$, $T_0 = Q - K_0 = (t_{ij}^0)_{i,j=1}^\infty$ and $T_n = Q^1 - K_n = (t_{ij}^n)_{i,j=1}^\infty$, $n = 1, 2, \ldots,$ where $(t_{ij}^n)_{i,j=1}^\infty$ is an upper triangular matrix with respect to a suitable ONB $\{e_j^n\}_{j=1}^\infty$ for all $n = 0, 1, 2, \ldots$. Furthermore, by using the semicontinuity of the spectrum, K_n can be chosen so that $t_{jj}^n = 0$ for all $j = 1, 2, \ldots,$ and for all $n = 0, 1, 2, \ldots$.

Let $T = \bigoplus_{n=0}^\infty T_n$; then $T = (T_{ij})_{i,j=1}^\infty$ with respect to the orthogonal direct sum $\mathcal{H}_\infty = \bigoplus_{l=1}^\infty \mathcal{H}_l$, where $\mathcal{H}_l$ is the closed linear span of the orthonormal system $\{e_l^n\}_{n=0}^\infty$ $(l = 1, 2, \ldots)$, $T_{ij} = 0$ if $i \geqslant j$ and T_{ij} is a diagonal (normal) operator with diagonal entries $t_{ij}^0, t_{ij}^1, t_{ij}^2, \ldots,$ if $i < j$. Clearly, $K = \bigoplus_{n=0}^\infty K_n \in \mathcal{K}(\mathcal{H}_\infty)$ and $Q_2 = T + K$.

Now we are in a position to apply the same argument as in the first part of the proof of Theorem 1.2 in [**4**]. Let P_m be the orthogonal onto $\bigoplus_{l=1}^m \mathcal{H}_l$ $(m = 1, 2, \ldots)$ and let τ be a faithful representation of the C^*-algebra $[C^*(T; P_1, P_2, \ldots) + \mathcal{K}(\mathcal{H}_\infty)]/\mathcal{K}(\mathcal{H}_\infty)$ in the separable Hilbert space $\mathcal{H}_\tau$, where $C^*(T; P_1, P_2, \ldots)$ is the separable C^*-algebra generated by T and the projections P_m, $m = 1, 2, \ldots$.

Let $S = \tau \circ \pi(T)$ and $R_m = \tau \circ \pi(P_m)$, $m = 1, 2, \ldots$. Since $P_m T P_m = P_m T$, it readily follows that $R_m S R_m = R_m S$ and therefore $R_m S R_m$ is the direct sum of an $m \times m$ upper triangular operator matrix with the 0 operator; furthermore, the upper triangular matrix has 0's *in the diagonal entries*; hence, $(R_m S R_m)^m = 0$ for all $m = 1, 2, \ldots$. On the other hand, since τ is faithful and $\pi(T)^{k-1} = \pi(Q_2)^{k-1} \neq 0$ for all $k = 1, 2, \ldots,$ given k there exists a minimal index $m(k)$ such that $(R_{m(k)}SR_{m(k)})^{k-1} \neq 0$. Then by [**14**, Theorem 1.3], there exists $Q_3 \approx Q \oplus S^{(\infty)}$ in $\mathcal{U}(Q)^-$ and, by Rota's corollary [**11**] and standard arguments (see, e.g., [**1**], [**8**]), $0^{(\infty)} \oplus (R_{m(k)}SR_{m(k)})^{(\infty)} \in \mathcal{S}(Q)^-$ for all $k = 1, 2, \ldots$.

Let $q_k \in \mathcal{L}(\mathbf{C}^k)$ be the truncated shift of order k (i.e., $q_k e_1 = 0$, $q_k e_j = e_{j-1}$ for all $j = 2, 3, \ldots, k$, with respect to the canonical ONB of $\mathbf{C}^k$). Since $\operatorname{Ran}[(R_{m(k)}SR_{m(k)})^{(\infty)}]^{k-1}$ contains an infinite dimensional (*closed*) subspace, it follows from [**1**, Lemma 2.1] that $q_k^{(\infty)} \oplus 0^{(\infty)} \in \mathcal{S}(Q)^-$ for all $k = 1, 2, \ldots,$ and therefore, by [**1**, Theorem 7.1], Q is a u.q.o. □

Recall that the similarity orbit of an element of $\mathcal{A}(\mathcal{H})$ is equal to $\mathcal{S}(a) = \{\tau(a)\colon \tau \in \text{Automorphisms of } \mathcal{A}(\mathcal{H})\} = \{waw^{-1}\colon w \text{ is invertible in } \mathcal{A}\}$ [**1**], [**12**] and that every nilpotent (quasinilpotent, resp.) in $\mathcal{A}(\mathcal{H})$ can be lifted to a nilpotent of the same order (quasinilpotent, resp.) in $\mathcal{L}(\mathcal{H})$ [**10**], [**13**]. By using these observations and Theorem 1, we can easily obtain the following nice

COROLLARY 1. *Let* $a \in \mathcal{A}(\mathcal{H})$; *then the following are equivalent*:

(i) $\mathcal{S}(a)^-$ *contains every quasinilpotent element of* $\mathcal{A}(\mathcal{H})$;

(ii) $\mathcal{S}(a)^-$ *contains every nilpotent element of* $\mathcal{A}(\mathcal{H})$;

(iii) $\mathcal{S}(a)^-$ *contains a sequence* $\{n_{j(k)}\}_{k=1}^{\infty}$ *of nilpotent elements of* $\mathcal{A}(\mathcal{H})$ *such that* $\lim(k \to \infty) j(k) = \infty$ *and* $n_{j(k)}^{j(k)} \neq 0$;

(iv) *a is quasinilpotent, but not nilpotent.*

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). The first two are trivial implications and the third one follows from [**8**, Proposition 1(i) and (vii)].

(iv) $\Rightarrow$ (i) If a satisfies (iv); then it can be lifted to an $A \in \mathbf{Q}_{ae}(\mathcal{H})$ [**10**], [**13**]. Since, by Theorem 1, $\mathcal{S}(\mathcal{A})^-$ contains every quasinilpotent of $\mathcal{L}(\mathcal{H})$, it follows from [**10**], [**13**] that $\mathcal{S}(a)^- = \mathcal{S}[\pi(\mathcal{A})]^- \supset \pi[\mathcal{S}(\mathcal{A})]^- \supset \{q\colon q \text{ is a quasinilpotent of } \mathcal{A}(\mathcal{H})\}$. □

3. The nonseparable case. Throughout this section, $\mathcal{H}$ will denote a nonseparable complex Hilbert space of (topological) dimension $h > \aleph_0$. Given a cardinal number α, $\aleph_0 \leqslant \alpha \leqslant h$, $\mathcal{I}_\alpha = \{T \in \mathcal{L}(\mathcal{H})\colon \dim(\operatorname{Ran} T)^- < \alpha\}$ is a bilateral ideal of $\mathcal{L}(\mathcal{H})$ and $\mathcal{J}_\alpha = \mathcal{I}_\alpha^-$ is a closed bilateral ideal (we shall simply say "$\mathcal{J}_\alpha$ is an ideal"). Furthermore, every ideal of $\mathcal{L}(\mathcal{H})$ is either trivial or equal to $\mathcal{J}_\alpha$ for some α, $\aleph_0 \leqslant \alpha \leqslant h$ [**3**].

A quasinilpotent $Q \in \mathcal{J}_\alpha$ is a u.q.o. for the ideal $\mathcal{J}_\alpha$ if $\mathcal{S}(Q)^-$ contains every quasinilpotent element of $\mathcal{J}_\alpha$. It is clear that if Q satisfies the above conditions, then Q^k cannot belong to any strictly smaller ideal ($k = 1, 2, \ldots$). The existence of some u.q.o. for the ideal $\mathcal{J}_\alpha$ strongly depends on the properties of α (see [**7**]). The best possible result, except perhaps for the ideal $\mathcal{K}$ of compact operators, is contained in Theorem 2 below. Recall that a cardinal number α is $\aleph_0$-irregular if $\alpha = \sup_n \alpha_n$ for a strictly increasing sequence $\{\alpha_n\}_{n=1}^{\infty}$ of cardinals [**3**]; then, we have

THEOREM 2. (i) $Q \in \mathcal{L}(\mathcal{H})$ *is a u.q.o. if and only if Q is quasinilpotent and* $Q^k \notin \mathcal{J}_h$, *the maximal ideal, for any* $k = 1, 2, \ldots$

(ii) *If either* $\alpha = \aleph_{\nu+1}$, *for some ordinal number* ν, *or* α *is* $\aleph_0$*-irregular and strictly larger than* $\aleph_0$, *then* $Q \in \mathcal{J}_\alpha$ *is a u.q.o. for* $\mathcal{J}_\alpha$ *if and only if* Q *is quasinilpotent and* $Q^k \notin \mathcal{J}_\beta$ *for any* $\beta < \alpha$ *and for any* $k = 1, 2, \ldots$.

(iii) *If* α *is* $\aleph_0$*-regular, but not of the form* $\aleph_{\nu+1}$ *for some ordinal number* ν, *then* $\mathcal{J}_\alpha$ *does not have a u.q.o.*

PROOF. (ii) Let $Q \in \mathcal{J}_\alpha$, where $\alpha > \aleph_0$ is an $\aleph_0$-irregular cardinal ($\alpha = \aleph_{\nu+1}$ for some ordinal number ν, resp.), be a quasinilpotent operator such that no power of Q belongs to a strictly smaller ideal. Then Q admits a decomposition $Q = 0^{(h')} \oplus \{\bigoplus Q_\gamma\colon \gamma \in \Gamma\}$, where Q_γ is a quasinilpotent operator acting on a separable infinite dimensional Hilbert space $\mathcal{H}_\gamma$, $Q_\gamma^{k-1} \notin \mathcal{K}(\mathcal{H}_\gamma)$ ($k = 1, 2, \ldots$), $\mathbf{C}(\Gamma) = \alpha$ ($\mathbf{C}(\Gamma) = \aleph_\nu$, resp.) and $h' = h$ or 0, according to $\alpha < h$ or $\alpha = h$, resp. ($h' = h$, resp.). Furthermore, given k, the cardinals of the subsets $\Gamma_{k,n} = \{\gamma \in \Gamma\colon \|Q_\gamma^{k-1}x\| \geqslant (1/n)\|x\|$ for all x in a suitable infinite dimensional subspace of $\mathcal{H}_\gamma\}$ satisfy $\mathbf{C}(\Gamma_{k,n}) < \alpha = \sup_n \mathbf{C}(\Gamma_{k,n})$ (there exists $\delta_k > 0$ such that $\|Q_\gamma^{k-1}x\| \geqslant \delta_k\|x\|$ for all x in a suitable infinite dimensional subspace of $\mathcal{H}_\gamma$, for all $\gamma \in \Gamma$, resp.). The details of the decomposition can be easily checked by using the results of **[3]**, **[9]**.

Thus, given $\beta < \alpha$, there exists an m such that $\mathbf{C}(\Gamma_{k,m}) \geqslant \beta$. Applying the arguments of the proof of Theorem 1 to Q_γ for all $\gamma \in \Gamma_{k,m}$ (for all $\gamma \in \Gamma$, resp.), we conclude that $q_k^{(\beta)} \oplus 0^{(h)} \in \mathcal{S}(Q)^-$ for all $\beta < \alpha$ and for all $k = 1, 2, \ldots$, and therefore Q is a u.q.o. for $\mathcal{J}_\alpha$ **[1]**.

Conversely, if Q is a u.q.o. for $\mathcal{J}_\alpha$, then it is clear that if $Q^k \in \mathcal{J}_\beta$ for some k and some $\beta < \alpha$, then $\mathcal{S}(Q)^- \subset \{A \in \mathcal{L}(\mathcal{H})\colon A^k \in \mathcal{J}_\beta\}$, a contradiction **[1]**, **[8]**.

(i) follows by a minor modification of the above proof for the case when $\alpha = \aleph_{\nu+1}$. Finally, (iii) is contained in **[7**, Theorem 3**]**. □

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