A-THEORY AND A-HOMOLOGY RELATIVE TO A
\(\text{II}_\infty\)-FACTOR

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Abstract. Let \(X\) be a compact space and \(M\) be a factor of type \(\text{II}_\infty\) acting on a separable Hilbert space. Let \(K_M(X)\) denote the Grothendieck group generated by the semigroup of isomorphism classes of \(M\)-vector bundles over \(X\), and, if \(X\) is also metric, let \(\text{Ext}^M(X)\) denote the group of equivalence classes of extensions of \(C(X)\) relative to \(M\). We show that \(K_M(X)\) is the direct sum of the even-dimensional Čech cohomology groups of \(X\), and that \(\text{Ext}^M(X)\) is the direct product of the odd-dimensional Čech homology groups of \(X\).

Introduction. Recently Brown, Douglas, and Fillmore [8] have constructed a generalised homology theory called \(K\)-homology, which, in a sense made rigorous in [8], is dual to \(K\)-theory. Their construction is in terms of extensions of commutative \(C^*\)-algebras by the ideal of compact operators on a separable Hilbert space. Fillmore [12] and Cho [9] have investigated the analogous construction with the compact operators replaced by the closed two-sided ideal generated by the finite projections in a factor of type \(\text{II}_\infty\). They have constructed (see [9]) a generalised homology theory \(\{\text{Ext}^M\}\) on the category of compact metric spaces, which we shall call \(K\)-homology relative to the \(\text{II}_\infty\)-factor \(M\). In [6] Breuer has considered a theory of vector bundles relative to \(M\) and introduced a functor \(K_M\) which has topological properties like those of \(K\)-theory. We shall construct a generalised cohomology theory \(\{K_M^*\}\) (\(K\)-theory relative to \(M\)) from Breuer's functor, identify it in terms of the conventional \(K\)-functor and show that \(K_M(X)\) is the direct sum of the even-dimensional real cohomology of \(X\) for any compact space \(X\). Then we shall deduce the corresponding result for \(\text{Ext}^M\); namely that \(\text{Ext}^M(X)\) is the direct product of the odd-dimensional real homology of \(X\). We mention that the results in this note all follow in standard fashion from the recent literature; our goal is merely to point out some interesting consequences of the work of Breuer [6] and Cho [9]. Along the way we provide a proof of Proposition 2, which has been stated and used by Singer in [18].

First we set up some notation. Throughout, all topological spaces will be Hausdorff, and \(M\) will be a factor of type \(\text{II}_\infty\) acting on a separable Hilbert space \(H\). We shall denote by \(P_J(M)\) the set of finite projections of \(M\) and by \(\dim: P_J(M) \to \mathbb{R}^+\) the Murray-von Neumann dimension function of \(M\). For
details on such matters, we refer to [10]. In addition, we shall write \( \mathcal{K}(M) \) for the closed two-sided ideal of \( M \) generated by \( P_j(M) \), \( \mathcal{A}(M) \) for the quotient algebra \( M/\mathcal{K}(M) \) and \( \mathcal{S}(M) \) for the set of operators which are Fredholm relative to \( M \) (cf. [5]). Our terminology as regards \( K \)-theory will be that of [1]. By a generalised (Čech) cohomology theory on compact pairs, we shall mean a sequence \( \{K^n\} \) of contravariant functors which satisfy the three axioms of continuity, excision and exactness (cf. [20, §1]). We observe that continuous functors are necessarily homotopy invariant [20, Theorem 2.1], so that such theories satisfy the first six of the Eilenberg-Steenrod axioms. We shall need the following lemma.

**Lemma.** If \( \mu: \{H^n\} \to \{K^n\} \) is a natural transformation between generalised cohomology theories such that \( \mu: H^n(X) \to K^n(X) \) is an isomorphism for all \( n \) when \( X \) is a point, then \( \mu \) is a natural equivalence.

**Proof.** That \( \mu \) is an equivalence on compact polyhedra follows from the argument of [19, Theorem 4.8.10]. But every compact space is the inverse limit of spaces with the homotopy type of compact polyhedra [19, Lemma 6.6.7], and so the result holds on the category of compact spaces.

1. Let \( X \) be a compact space. Breuer [6] introduced the notion of an \( M \)-vector bundle over \( X \)--namely, a Hilbert space bundle over \( X \) whose transition functions take values in \( M \) and whose fibres are of the form \( E(H) \) for some \( E \) belonging to \( P_j(M) \). The set \( \text{Vect}_M(X) \) of \( M \)-isomorphism classes of \( M \)-vector bundles over \( X \) is a semigroup under direct sum; if \( f \) is a continuous map from \( Y \) to \( X \), then \( f \) induces (via pull-back of bundles) a semigroup homomorphism \( f^*: \text{Vect}_M(X) \to \text{Vect}_M(Y) \). If we denote the Grothendieck group of \( \text{Vect}_M(X) \) by \( K_M(X) \), then \( K_M \) is a contravariant functor from compact spaces to abelian groups. If \( X \) is a compact space with distinguished base point \( x_0 \) and \( \iota: \{x_0\} \to X \) is the inclusion, then we write \( K_M(X) \) for the kernel of the map \( i^*: K_M(X) \to K_M(\{x_0\}) \). Breuer proved that \( K_M \) is homotopy invariant, and that \( K_M(X) \) is a module over the ring \( K(X) \); it is easy to check from the definition [6, p. 417] that this module action is natural. The main result of Breuer’s article is the periodicity theorem for \( K_M \); namely that for any locally compact space \( X \), \( K_M(\mathbb{R}^2 \times X) \cong K_M(X) \), where for \( Y \) locally compact \( K_M(Y) \) stands for the reduced group \( \tilde{K}_M(Y \cup \{\infty\}) \) of the one point compactification of \( Y \). This isomorphism is natural since the inverse \( \beta_X \) is defined in terms of the module action.

We define \( K_M^n(X) = K_M(\mathbb{R}^n \times X) \) (for \( n > 0 \)) and, inductively, \( K_M^n(X) = K_M^{n-2}(X) \) for positive \( n \). If for a compact pair \((X, Y)\) we now set \( K_M^n(X, Y) = \tilde{K}_M^n(X/Y) \) (the base point is \( Y/Y \)) then \( \{K_M^n\} \) is a sequence of contravariant functors from compact pairs to abelian groups.

**Proposition 1.** \( \{K_M^n\} \) is a generalised cohomology theory on compact pairs.

**Proof.** That \( \{K_M^n\} \) satisfies excision is obvious. To verify continuity and exactness we shall use the theorem of Breuer that \( K_M(X) \cong [X, \mathcal{S}(M)] \) (see
[6, Theorem 1, p. 414]); an inspection of Breuer’s construction yields that the isomorphism is natural. Since \( \mathcal{T}(M) \) is an open set in the Banach space \( M \) ([5, II, Corollary 2 to Theorem 1]), \( \mathcal{T}(M) \) and its loop spaces \( \Omega^n \mathcal{T}(M) \) are ANR’s (cf. [14, Chapter 1]). It follows from the periodicity theorem that \( \pi_n(\mathcal{T}(M)) \cong \pi_n(\Omega^n \mathcal{T}(M)) \) for every \( n > 0 \), and so \( \mathcal{T}(M) \) and \( \Omega^n \mathcal{T}(M) \) are homotopy equivalent by [17, Theorem 15]. Thus \( \{K^n_M\} \) is given by a spectrum, and so by [21, §5] satisfies the exactness axiom on finite complexes. We can deduce that \( K_M \) is continuous from the fact that \( \mathcal{T}(M) \) is an ANR, and the result follows.

If \( X \) is a compact space, \( r \in \mathbb{R}^+ \) and \( E \in P_f(M) \) satisfies \( \dim E = r \), then, as in the construction of the module action [6, p. 417], there is a map \( \lambda_r: \text{Vect}(X) \to \text{Vect}_M(X) \) given by \( \lambda_r(a) = a \otimes (X \times E(H)) \). Thus there is a pairing \( (a, r) \to \lambda_r(a): \text{Vect}(X) \otimes \mathbb{R}^+ \to \text{Vect}_M(X) \) which induces a natural transformation \( K(\cdot) \otimes \mathbb{R} \to K_M(\cdot) \). We observe that \( \lambda: K(X) \otimes \mathbb{R} \to K_M(X) \) is an isomorphism when \( X \) is a one point space.

**Proposition 2 (Singer).** The functors \( K(\cdot) \otimes \mathbb{R} \) and \( K_M(\cdot) \) are naturally equivalent (via \( \lambda \)) on the category of compact spaces. In particular, \( K_M(\cdot) \) is independent of the factor \( M \).

**Proof.** The functors \( K^* \) form a generalised cohomology theory, and this implies that \( K^*(\cdot) \otimes \mathbb{R} \) do also. For clearly \( K^n(\cdot) \otimes \mathbb{R} \) is a sequence of contravariant functors satisfying the excision axiom; the exactness axiom for \( K(\cdot) \otimes \mathbb{R} \) follows since \( \mathbb{R} \) is torsion-free and the continuity axiom follows since tensoring with \( \mathbb{R} \) commutes with direct limits [3, pp. 33–34]. Let \( X \) be a compact space and let \( \text{Per}: K(X) \to K(\mathbb{R}^2 \times X) \) and \( \text{Per}_M: K_M(X) \to K_M(\mathbb{R}^2 \times X) \) denote the periodicity maps of \( K \)-theory and \( K_M \)-theory respectively. Then \( \text{Per} \) is given by taking the external product with the Bott element \( b \in K(\mathbb{R}^2) \) [4, p. 118], and \( \text{Per}_M \) is the analogous external product for \( K_M \)-theory with the same element \( b \in K(\mathbb{R}^2) \) [6, p. 426]. It follows from elementary properties of the external product (cf. [6, §4.11]) that the diagram

\[
K(X) \otimes \mathbb{R} \xrightarrow{\text{Per} \otimes \text{id}} K(\mathbb{R}^2 \times X) \otimes \mathbb{R}
\]

commutes. Hence \( \lambda \) can be extended to give a natural transformation between the generalised cohomology theories \( K^*(\cdot) \otimes \mathbb{R} \) and \( K_M^*(\cdot) \). As observed above \( \lambda: K^n(\text{pt}) \otimes \mathbb{R} \to K^n_M(\text{pt}) \) is an isomorphism when \( n = 0 \); since every \( M \)-vector bundle on \( S^1 \) is trivial [6, Corollary 2, p. 404] it is also an isomorphism for \( n = -1 \), and it follows that \( \lambda \) is an isomorphism for all \( n \in \mathbb{Z} \). The results now follow from the lemma in the introduction.

It is a standard result in \( K \)-theory that for a compact space \( X \), \( K(X) \otimes \mathbb{R} \) is the direct sum of all the groups \( H^p(X; \mathbb{R}) \) for \( p \) even, where \( H^p(X; \mathbb{R}) \) denotes the \( p \)th Čech cohomology group of \( X \) with real coefficients. (This is a consequence of [2, p. 19] and the universal coefficient theorem. A more
elementary proof is contained in [1, §3.2]; here, however, we have to invoke
the Eilenberg-Steenrod uniqueness theorem to deduce that the \( H^p \)'s are in
fact Čech cohomology.) It now follows immediately from Proposition 2 that:

**Corollary 3.** For any compact space \( X \) there is a natural isomorphism

\[
K_M(X) \cong \bigoplus \{ H^p(X; \mathbb{R}) : p \text{ even}, p > 0 \}.
\]

2. Let \( X \) be a compact metric space. An extension of \( C(X) \) relative to \( M \) is
a unital *-monomorphism \( \tau: C(X) \to \mathcal{A}(M) \). Two such extensions \( \tau_1, \tau_2 \) are
equivalent if there is an inner automorphism \( \alpha \) of \( M \) (which maps \( \mathcal{K}(M) \) onto
\( \mathcal{K}(M) \) and so induces an automorphism \( \alpha \) of \( \mathcal{A}(M) \)) with \( \tau_2 = \alpha \circ \tau_1 \). The set
\( \text{Ext}^M(X) \) of equivalence classes of extensions of \( C(X) \) relative to \( M \) is a
group (see [12]), is a homotopy invariant functor of the space \( X \) and can be
used to define a generalised homology theory (see [9]). Cho also proves in [9]
that \( \text{Ext}^M \) is naturally equivalent to \( \text{Hom}(\mathcal{K}(\mathcal{S}(\cdot)), \mathbb{R}) \)--and so is independent
of \( M \).

**Proposition 4.** For any compact metric space \( X \) there is a natural
isomorphism

\[
\text{Ext}^M(X) \cong \prod \{ H_p(X; \mathbb{R}) : p \text{ odd}, p \geq 1 \}
\]

where \( H_p(X, \mathbb{R}) \) denotes the \( p \)th Čech homology group of \( X \) with real
coefficients.

**Proof.** First we suppose that \( X \) is a compact polyhedron. By the main
theorem of [9], \( \text{Ext}^M(X) \cong \text{Hom}(\mathcal{K}(SX); \mathbb{R}) \) where \( SX \) denotes the unre-
duced suspension of \( X \). This in turn can be identified with
\( \text{Hom}_{\mathbb{R}}(\mathcal{K}(SX) \otimes \mathbb{Z}, \mathbb{R}) \), which is isomorphic to \( \text{Hom}_{\mathbb{R}}(\mathbb{Z} \oplus \bigoplus_p \text{H}^p(SX; \mathbb{R}), \mathbb{R}) \)
by Corollary 3. Since \( X \) is a compact polyhedron \( \text{Hom}(\mathcal{H}^p(SX); \mathbb{R}) \cong
\mathcal{H}_p(SX) \) [13, 23.14] and so \( \text{Ext}^M(X) \cong \prod_p \text{H}_p(SX; \mathbb{R}) \). But \( \mathcal{H}_p(SX) \cong
\text{H}_{p-1}(X) \), and we have the result for compact polyhedra. The general case
now follows by observing that both \( \text{Ext}^M \) and \( \prod \text{H}_p(\cdot, \mathbb{R}) \) are continuous
functors [9, Corollary 1].

**Remarks.** Although the Čech homology theory \( H_\ast(\cdot, G) \) with coefficients
in an abelian group \( G \) satisfies the continuity axiom, it does not in general
have a long exact sequence; the appropriate theory for compact metric spaces
is Steenrod homology, denoted \( \ast H_\ast(\cdot, G) \). In addition to the seven Eilenberg-
Steenrod axioms, \( \ast H_\ast \) satisfies the relative homeomorphism axiom and the
cluster axiom (see [15] or [16]), and is characterised uniquely by these axioms
[16, Theorem 3]. For an arbitrary coefficient group Čech homology satisfies
all these axioms except exactness; however, when the coefficient group is \( \mathbb{R} \),
Čech homology is exact [11, Theorem IX.7.6] and so coincides with Steenrod
homology on compact metric spaces. Thus the last proposition is valid with the
Čech groups \( H_\ast(X; \mathbb{R}) \) replaced by the corresponding Steenrod groups.
For further details on the relationships between Čech and Steenrod homology we refer to [15] and [16]; the Čech theory is discussed in detail in [11].

In [9] Cho proves the Ext_* is a generalised Steenrod theory which is also continuous; hence Ext_# is also a generalised Čech theory. (Here by a generalised theory we mean one which satisfies all the appropriate axioms except dimension; the axioms for Čech homology are given in [11, Chapter X].) This is not the case for the Brown-Douglas-Fillmore theory Ext_*; it is a generalised Steenrod theory but is not continuous— in fact Brown [7] has shown that it fails to be continuous in the same way as the Steenrod homology theory. This is discussed in [15].

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REFERENCES


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