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THE UNIFORM CONTINUITY OF CERTAIN TRANSLATION SEMIGROUPS[1]

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ABSTRACT. Let $S_h f(x) = f(x + h)$ for $h \geqslant 0$, for $f \in L^2(R^+; K)$, where K is a separable Hilbert space. The translation semigroup S_h when restricted to an invariant subspace L is uniformly continuous if and only if G_L is an inner function and has an analytic continuation across an open arc of the unit circle at $z = 1$. The operator-valued function G_L is associated with the invariant subspace L by Beurling's theorem.

If a function in the Lebesgue space of square-integrable functions on the positive real axis is composed with a translation of its domain by a positive amount, the composite function remains in that Lebesgue space. The consideration of all such translations gives rise to a semigroup of translation operators. This semigroup is not uniformly continuous, but may become so when restricted to an invariant subspace. Necessary and sufficient conditions for uniform continuity are given in the general case that the functions take values in a separable Hilbert space. The conditions are given both in terms of the spectrum of the infinitesimal generator of the semigroup and in terms of a representation using operator valued inner functions.

1. Translation semigroups. Let K be a separable Hilbert space, and let f be a function defined on R^+, the nonnegative reals, with values in K. The function f is said to be in $L^2(R^+; K)$ if $\langle f(x), k \rangle$ is measurable for every $k \in K$ and

$$\|f\|^2 = \int_0^\infty \|f(x)\|^2 \, dx < \infty.$$

Following Helson [4], $\|f(x)\|$ denotes the norm of the vector $f(x)$ in K, and $\|f\|$ denotes the corresponding norm of f in $L^2(R^+; K)$. Naturally, two functions f and g are equal in $L^2(R^+; K)$ if and only if $f(x) = g(x)$ a.e.

Consider the following operator S_h:

$$(S_h f)(x) = f(x + h) \quad \text{where } x \geqslant 0, h \geqslant 0.$$

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where $f \in L^2(R^+; K)$. Then $S_h f$ is also in $L^2(R^+; K)$. This collection S_h, $h > 0$, of operators forms a strongly continuous semigroup **[5]**. Suppose a closed linear subspace L of $L^2(R^+; K)$ is invariant under the semigroup S_h, i.e.

$$S_h(L) \subseteq L \quad \text{for every } h \geqslant 0.$$

We shall denote by S_h^L the restricted semigroup defined by:

$$S_h^L = P_L S_h|_L$$

where P_L denotes the orthogonal projection onto L. Then we ask: For which invariant subspaces L is the semigroup S_h^L found to be uniformly continuous? The object of this paper is to answer this question by representing such invariant subspaces L using an operator valued analytic function, which will be developed in the next section.

2. The representation theorem. A function f which is analytic in the open disk U with values in K is said to be in $H^2(U; K)$ if it satisfies:

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left\| f(re^{i\theta}) \right\|^2 d\theta < \infty.$$

The radial limits exist, a.e., defining the boundary value function:

$$f(e^{i\theta}) = \lim f(re^{i\theta}) \quad \text{as } r \to 1^-.$$

These boundary value functions give the inner product for $H^2(U; K)$:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{i\theta}), g(e^{i\theta}) \rangle \, d\theta$$

where $\left\langle f(e^{i\theta}), g(e^{i\theta}) \right\rangle$ denotes the inner product in K.

For a function f in $H^2(U; K)$, the backward shift T applied to f yields:

$$(Tf)(z) = (f(z) - f(0))/z$$

which is again in $H^2(U; K)$. If L is a closed linear subspace space of $H^2(U; K)$ such that $T(L) \subseteq L$, then L is said to be invariant under T.

A partial isometry A is a linear transformation that is isometric on the orthogonal complement of its kernel, called the initial space of A. Then A^*A is a projection operator, projecting orthogonally onto the initial space of A.

REPRESENTATION THEOREM ([1], [3], [6]). *If $L \subseteq H^2(U; K)$ is invariant under the backward shift T, then there exists an analytic operator valued function G such that*

$$H^2(U; K) = L \oplus GH^2(U; K)$$

where G satisfies the following:

(i) *for $|z| < 1$, $G(z)\colon K \to K$ is a contraction.*

(ii) *$G(e^{i\theta}) = \lim G(re^{i\theta})$ exists, a.e., as $r \to 1^-$, and $G(e^{i\theta})\colon K \to K$ is a partial isometry, a.e., with the same initial space for all such θ.*

G is determined by L up to composition on the right by a partial isometry corresponding to a different choice of an isometrically isomorphic initial space.

3. Conditions for uniform continuity. First recall some useful properties of the Laguerre functions [**10**]. For $x \in R^+$, the nth Laguerre function g_n is defined by:

$$g_n(x) = \frac{1}{n!} \exp(x/2) \frac{d^n}{dx^n} x^n \exp(-x)$$

for $n = 0, 1, 2, 3 \ldots$. It is well known that these functions form an orthonormal basis for $L^2(R^+; C)$. We shall also need the following property:

LEMMA. *For each n, the closed linear subspace spanned by $g_0, g_1, \ldots, g_n$ is invariant under differentiation. The following formula involving the Fourier transform is also useful*:

LEMMA.

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(isx)\, g_n(x) dx = \frac{2(2is+1)^n}{2\pi(2is-1)^{n+1}}, \qquad n = 0, 1, 2, \ldots.$$

Next we define a mapping J: $L^2(R^+; K) \to H^2(U; K)$. Consider the orthonormal basis for $L^2(R^+; K)$: $g_n k_\alpha$: $n = 0, 1, 2, \ldots$; $\alpha = 1, 2, 3, \ldots$, where k_α is from a fixed orthonormal basis for K. We define:

$$J(g_n k_\alpha)(z) = z^n k_\alpha \quad \text{for } |z| < 1.$$

Then J extends to a continuous linear isometry onto $H^2(U; K)$. Now we consider the image $J(L)$ of the invariant subspace L for the semigroup S_h.

THEOREM. *$J(L)$ is invariant under T.*

PROOF. First we recall the important result of Paley and Wiener.

THE ONE-SIDED PALEY-WIENER THEOREM. *The Fourier transform F is an isometry on $L^2(R^+; K)$ with image $H^2(\Pi; K)$, i.e. the Hardy space of functions analytic in the upper half plane Π with*

$$\sup_{t>0} \int_{-\infty}^{\infty} \|f(s+it)\|^2 \, ds < \infty.$$

Now we factor the Fourier transform through $H^2(U; K)$ as follows: W: $H^2(U; K) \to H^2(\Pi; K)$ is defined by

$$(Wf)(\beta) = 2f\left(\frac{2i\beta+1}{2i\beta-1}\right) / \left(\sqrt{2\pi}\,(2i\beta - 1)\right), \qquad \operatorname{im} \beta > 0.$$

W is seen to be isometric and even has an inverse:

$$(W^{-1}g)(z) = \sqrt{2\pi}\; g\left(\frac{z+1}{2i(z-1)}\right) / (z-1), \qquad |z| < 1.$$

But the most important fact about W is that:

$$WJ(g_n k_\alpha)(\beta) = F(g_n k_\alpha)(\beta)$$

using the above results about Laguerre functions. Hence we have that $WJ = F$ on $L^2(R^+; K)$. Note the following:

LEMMA. *The function $h(s) = (2is + 1)(2is - 1)^{-1}$ can be uniformly approximated on every compact subset of R by a sequence of trigonometric polynomials of the form*:

$$t_n(s) = \sum_{k=0}^{N_n} c_{nk}\exp(ib_{nk}s)$$

where $b_{nk} \geqslant 0$ and $|t_n(s)| < M$ for all $s \in R$.

Now suppose that N is a closed linear subspace of $H^2(\Pi; K)$ which is invariant under multiplication by $\exp(ias)$ for all $a \geqslant 0$, hence under multiplication by any trigonometric polynomial with nonnegative exponents. Using the above lemma, we define:

$$(V_n g)(\beta) = t_n(\beta)g(\beta)$$

for $g \in N$ and im $\beta > 0$. These operators V_n converge strongly to V_0 given by:

$$(V_0 g)(\beta) = \left(\frac{2i\beta + 1}{2i\beta - 1}\right)g(\beta).$$

Since N is closed, $V_0(N) \subseteq N$. Now $W^{-1}(N) \subseteq H^2(U; K)$ is a closed linear subspace on which we set $V = W^{-1}V_0W$. Then on $W^{-1}(N)$ we have $(Vf)(z) = zf(z)$, so that $W^{-1}(N)$ is invariant under T^*, the adjoint of T.

Now we simply choose $N =$ the orthogonal complement of $F(L)$. If g is orthogonal to $F(L)$, $\langle e^{ih\beta}g, F(f)\rangle = \langle g, e^{-ih\beta}F(f)\rangle$ which is equal to zero, since $F(S_h f) = e^{-ih\beta}F(f)$, which is the Fourier transform of an element of L, since L is invariant under S_h. Finally $W^{-1}(N \oplus F(L)) = W^{-1}(N) \oplus J(L)$, hence $J(L)$ is invariant under T, the backward shift.

Therefore by the representation theorem we have that:

$$H^2(U; K) = J(L) \oplus G_L H^2(U; K)$$

where G_L has the properties mentioned above. If G_L has unitary boundary values a.e., then G_L is called an inner function.

THEOREM. *If G_L is inner, then the semigroup S_h^L is uniformly continuous if and only if G_L has an analytic continuation across an arc of the unit circle containing $z = 1$ in its interior.*

THEOREM. If G_L fails to be inner, then the semigroup S_h^L fails to be uniformly continuous.

4. Proof of the main result. The first part of the proof will show why the point $z = 1$ is so distinguished. Let the domain of D_L consist of all functions $f \in L$ such that this limit exists a.e. in the norm of K:

$$(D_L f)(x) = \lim(f(x + h) - f(x))/h \quad \text{as } h \to 0^+.$$

Then the operator D_L is the infinitesimal generator of the semigroup S_h^L. The semigroup is uniformly continuous if and only if D_L is bounded as an operator on L. At this point, we recall that the closed linear span of the

Laguerre functions is invariant under differentiation. Hence the closed linear span of $g_j k_\alpha$: $j = 0, 1, \ldots, n$; $\alpha = 1, 2, \ldots, m$, is invariant under the infinitesimal generator D_L. Therefore the composition JD_LJ^{-1} maps the closed linear span of $z^j k_\alpha$: $j = 0, 1, \ldots, n$; $\alpha = 1, 2, \ldots, m$, back into itself for each n and m.

LEMMA. *As unbounded operators on $J(L)$ we have*

$$2JD_LJ^{-1} = (T + I)(T - I)^{-1}.$$

The lemma follows from a comparison of $T + I$ and $2JD_LJ^{-1}(T - I)$ on each $z^j k_\alpha$. Since J is unitary, we conclude that D_L is bounded on L if and only if $(T - I)^{-1}$ is a bounded operator on $J(L)$, that is, if and only if the point $z = 1$ is in the resolvent set of T as an operator on $J(L)$.

So next we consider the relation of the resolvent set of T restricted to the invariant subspace $J(L)$ in terms of the possibility of analytically continuing the function G_L across arcs of the unit circle. First we shall restrict our attention to the case in which G_L is an inner function.

LEMMA. *Let $|\beta| \leqslant 1$. If the range of $G_L(\beta)$ contains $f(\beta)$ for each $f \in J(L)$, then the range of $G_L(\beta)$ is all of K.*

PROOF. Let $k \in K$. Define $p(z) = k$ for all $z \in U$. Then p is in $H^2(U; K)$. Decompose $p = f + G_L g$, with $f \in J(L)$ and $g \in H^2$. Evaluating at β, $k = p(\beta) = f(\beta) + G_L(\beta)g(\beta) \in$ range of $G_L(\beta)$.

LEMMA. *Let $|\beta| = 1, f \in J(L)$. If either β^* is in the resolvent of $T|J(L)$ or G_L is analytic at β, then f can be analytically continued across the unit circle at the point β.*

PROOF. Suppose β^* is in the resolvent of $T|J(L)$. For $f \in J(L)$ consider $f_w = (wT - I)^{-1}f$. For each w^{-1} in the resolvent, $f_w \in J(L)$ and depends analytically on w. The leading coefficient $a_0(z)$ for $f_w(z)$ provides analytic continuation to all points w with w^{-1} in the resolvent, in particular for β. Also we note that $\|f_w\| < \|(wT - I)^{-1}\| \cdot \|f\|$.

Now suppose G_L is analytic at β. Then $h(e^{i\theta}) = G_L^*(e^{i\theta})f(e^{i\theta})$ is the boundary value of a function analytic outside the closed unit disk in a region whose boundary curve includes an arc containing β. Since $f = G_L G_L^* f = Gh$ on that arc, Gh provides an analytic continuation of f across the unit circle at β.

Now let T_L^* denote the adjoint of T restricted to $J(L)$.

LEMMA. *$T_L^* f = T^* f + G_L k$, for each $f \in J(L)$, where k is a vector in K, depending on f.*

PROOF. $(T_L^* - T^*)f$ is orthogonal to $J(L)$, hence has the form Gh for some $h \in H^2(U; K)$. But the boundary value function of $G_L^*(T_L^* - T^*)f$ can only have a nonzero Fourier coefficient of order zero. Hence h is a constant function.

THEOREM. *If* $|\beta| < 1$, *then* β *is in the resolvent of* T *restricted to* $J(L)$ *if and only if* $G(\beta^*)$ *is invertible as an operator on* K.

PROOF. We actually consider the resolvent of T_L^*. Using the lemma relating the two adjoints, we consider the solution of

$$zf(z) - \beta f(z) = g(z) + G(z)k.$$

If the formal solution for $f(z)$ actually defines an element of $H^2(U; K)$, then both $G^*(e^{i\theta})f(e^{i\theta})$ and $e^{i\theta}G^*(e^{i\theta})f(e^{i\theta})$ can be continued analytically into the exterior of the closed unit disk. Hence $f \in J(L)$.

But $f \in H^2(U; K)$ whenever $g(\beta) + G(\beta)k = 0$. This occurs for a unique k provided that $G(\beta)$ is invertible. Conversely, if $\beta \in$ resolvent of T_L^*, then $G(\beta)$ is one-one and the range of $G(\beta)$ contains all the vectors $g(\beta)$ for $g \in J(L)$. By the lemma, the range of $G(\beta)$ is all of K, therefore $G(\beta)$ is invertible. *Scalar versions of these theorems appear in* **[2]**, **[8]**.

THEOREM. *If* $|\beta| = 1$, *then* β *is in the resolvent of* T *restricted to* $J(L)$ *if and only if* $G(z)$ *can be analytically continued across* $z = \beta^*$.

PROOF. First suppose that $\beta \in$ the resolvent of T_L^*. Then the function $T(G(z)k)$ is in $J(L)$ for any vector $k \in K$. By the lemma, $T(G(z)k)$ can be analytically continued across an arc of the unit circle at $z = \beta$, hence so can $G(z)k$. Furthermore,

$$\|G(z)\| < |z| \, \|(zT - I)^{-1}\| \, \|T(G(z))k\|,$$

so that G also can be analytically continued across $z = \beta$.

On the other hand, suppose G can be analytically continued at $z = \beta$. Then all the functions $f \in J(L)$ are analytic at β. Hence the argument for $|\beta| < 1$ can be repeated and for a given $g \in J(L)$ there exists a unique $k \in K$ such that $g(\beta) + G(\beta)k = 0$.

This completes the proof in the case that G is an inner function. In the case that G is noninner, we proceed as before and assert that the formual solution $f(z)$ is in $H^2(U; K)$ if and only if we have $G(\beta)h(\beta) + g(\beta) = 0$. If the boundary value functions of G are nonisometric partial isometries, then the projection of h onto the initial space M of G is constant, but this fails to uniquely determine h, since $M \neq K$. On the other hand, if the boundary value functions of G are nonunitary isometries whose initial space is K, then the lemma relating the two adjoints still holds and we can again consider

$$zf(z) - f(z) = g(z) + G(z)k$$

where $k \in K$ is a constant vector. Then the requirement that $g(\beta) + G(\beta)k = 0$ implies that $g(\beta) \in$ the range of $G(\beta)$. But since g is an arbitrary function in $J(L)$ this implies, by a previous lemma, that the range of $G(\beta)$ is all of K, which is false.

Hence if G_L fails to be an inner function, $\beta = 1$ is in the spectrum of T restricted to $J(L)$ and the semigroup S_h^L fails to be uniformly continuous.

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