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PARTITIONS INTO CHAINS OF A CLASS OF PARTIALLY ORDERED SETS

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ABSTRACT. Let a cube of side k in $\mathbf{R}^n$ be dissected into k^n unit cubes. The collection of all affine subspaces of $\mathbf{R}^n$ determined by the faces of the unit cubes forms a lattice $L(n, k)$ when ordered by inclusion. We explicitly construct a Dilworth partition into chains of $L(n, k)$.

A well-known result of Dilworth states that a finite partially ordered set can be partitioned into a number of chains equal to the maximum size of an antichain. The known proofs however provide no clue as to a specific construction of such a partition. It has been noted that a construction of such a partition in certain special cases leads to stronger conclusions in extremal set-theoretic problems, for example in Greene and Kleitman ([3] in bibliography), who obtain a Dilworth partition–as we shall call it–for the lattice of faces of the n-simplex, or Boolean algebra. This partition was also done by de Bruijn, Tengbergen, and Kruyswijk [1], and C. Leeb (unpublished).

More recently, Metropolis and Rota [6] have constructed a Dilworth partition of the lattice of faces of the n-cube. We now generalize both of the above constructions. We dissect a cube of side k in dimension n into k^n unit cubes, and take the resulting lattice of all linear subspaces determined by the faces of the k^n cubes. We construct a Dilworth partition of this lattice. The notation we use is adapted from Metropolis and Rota.

We first consider the $(k + 1)n$ hyperplanes in n-space defined by the equations $x_i = j$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k$. The nonempty intersections of such hyperplanes define the nonnull faces of the dissected n-cube of side k. Let $L(n, k)$ denote the collection of such faces ordered by inclusion, together with a minimum element 0 and maximum element 1. Then $L(n, k)$ is a lattice.

Next we consider the n-set $\{1, 2, \ldots, n\} = N$, and a sequence $A = (A_0, A_1, \ldots, A_k)$ such that $A_i \subseteq N$ for all i, and $A_i \cap A_j = \varnothing$ if $i \neq j$. We order such sequences by $A \leqslant B$ whenever $A_i \supseteq B_i$ for all i. We adjoin a minimum element 0, and let $O(n, k)$ denote the resulting partially ordered set. Then $O(n, k) \cong L(n, k)$, with the isomorphism defined by assigning to

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$(A_0, A_1, \ldots, A_k)$ the face of the dissected cube determined by the collection of all hyperplanes $x_i = j$ such that $i \in A_j$.

There is yet a third description of $L(n, k)$, up to isomorphism. Let $P = \{0, 1, 2, \ldots, k, x\}$ be a partially ordered set with $x > i$ for all i, $0 \leqslant i \leqslant k$, and i and j unrelated if $i \neq j$, $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant k$. Let $P(n, k)$ denote the direct product P^n with a minimum element 0 adjoined. Then $P(n, k) \cong L(n, k)$, with the isomorphism defined by assigning to $(p_1, \ldots, p_n) \in P^n$ the intersection of hyperplanes $x_i = j$ such that $p_i = j$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant k$. We note that in $P(n, k)$, $(p_1, \ldots, p_n) \leqslant (q_1, \ldots, q_n)$ whenever q_i equals either p_i or x for all i. We say p_i is a *significant symbol* if $p_i \in P - \{x\}$, and *nonsignificant* if $p_i = x$. The dimension of the face of the dissected cube corresponding to $(p_1, \ldots, p_n)$ is the number of nonsignificant symbols in $(p_1, \ldots, p_n)$, which we will also call the *dimension* of $(p_1, \ldots, p_n)$. We will also let the dimension of the element 0 be -1.

It is easy to check (by Theorem 4.1 or Theorem 4.10 of [3]) that $P(n, k)$ has the Lubell-Yamamoto-Meschalkin (LYM) property, namely, if F is an arbitrary anti-chain of $P(n, k)$, f_d is the number of elements of F of dimension d, and N_d is the dth Whitney number of $P(n, k)$ (i.e., N_d is the number of elements of $P(n, k)$ of dimension d), then

$$\sum_{d=-1}^{n} f_d / N_d \leqslant 1.$$

The LYM property implies the Sperner property, namely that the maximum size of an anti-chain is the maximum Whitney number.

We now note that for $d \geqslant 0$, $N_d = \binom{n}{d}(k+1)^{n-d}$. The following Lemma is proved by straightforward computation.

LEMMA. *A value of d which maximizes $\binom{n}{d}(k+1)^{n-d}$ is $[n/(k+2)]$, where brackets denote the greatest integer function.*

We now wish to decompose $P(n, k)$ into a disjoint union of chains, using the minimum possible number of chains. By Dilworth's Theorem, the number of chains required is equal to the cardinality of the largest anti-chain of $P(n, k)$, namely $\binom{n}{e}(k+1)^{n-e}$, where $e = [n/(k+2)]$. Clearly, each chain must contain precisely one element of $P(n, k)$ of dimension e. We describe an algorithm for such a Dilworth decomposition.

We begin with a subroutine $R(m)$ which takes a sequence of m significant symbols, $m \leqslant k + 1$, and replaces one symbol by x. Given $(a_1, a_2, \ldots, a_m)$, let $l \equiv \sum_{i=1}^{m} a_i \pmod{k+1}$, $1 \leqslant l \leqslant k + 1$. If $l \leqslant m$, replace a_l by x. Note that if $m < k + 1$ the subroutine will fail when $l > m$ and be terminated.

We now construct an ascending chain from each atom of $P(n, k)$. An atom is a sequence $(p_1, \ldots, p_n)$ of symbols, all significant. Apply $R(k + 1)$ to the last $k + 1$ symbols (or $R(n)$ to the entire sequence if $n < k + 1$), then in the resulting sequence apply $R(k+1)$ to the last $k+1$ symbols preceding the x, and continue repeatedly applying $R(k+1)$ to the last $k+1$ symbols preceding the first x. If only m significant symbols remain preceding the first x, with $m < k+1$, apply $R(m)$ to the first m symbols, and repeat until $m = 0$ or the subroutine fails.

A given atom, together with each of the sequences derived from it in turn by application of the subroutine, forms a chain in $P(n, k)$. Furthermore, we have assigned to a unique such chain every sequence $(p_1, \ldots, p_m)$ in $P(n, k)$ not containing a contiguous subsequence $(x, a_1, \ldots, a_{k+1})$ with $a_1, \ldots, a_{k+1}$ significant symbols. To see this, we show that the above process is reversible, that is, we can recover the atom whose chain contains $(p_1, \ldots, p_n)$. If $(p_1, \ldots, p_n)$ has no x we are done. If it has one x, let A denote the sequence of the last $k+1$ symbols (or n symbols if $n < k+1$). If $(p_1, \ldots, p_n)$ has at least two symbols x, let A denote the sequence of the last $k+1$ symbols preceding the second x (or all the symbols preceding the second x if there are fewer than $k+1$). Then A must contain exactly one x, say $A = (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_m)$. Replace x by $j - (\sum_{i=1}^{j-1} a_i) - (\sum_{i=j+1}^{m} a_i)$ (mod $k+1$). This replaces one x in $(p_1, \ldots, p_n)$ by a significant symbol, and iteration leads to the desired atom, as is easily verified.

The element 0 of $P(n, k)$ may be assigned arbitrarily to one of the chains, say, the one containing the vertex $(0, 0, \ldots, 0)$.

There remain unassigned to chains only those sequences containing contiguous subsequences of the form $(x, a_1, \ldots, a_{k+1})$ with $a_1, \ldots, a_{k+1}$ significant. We apply a bracketing algorithm as in [**6**]. Place brackets around every such contiguous subsequence. Then, ignoring the bracketed portion, again bracket the remaining sequence. Continue until no further bracketing is possible, in which case we say the sequence is completely bracketed. For example, if $k = 2$,

1x2x0210x1102x1

becomes

1(x2(x021)0(x110)2)x1

when completely bracketed.

We now take all completely bracketed sequences having a specified bracket placement and specified bracketed symbols. We then apply the above subroutine to the unbracketed symbols as before, thus assigning the completely bracketed sequences to chains. For example, since

xx1
1x1
101

is a chain, we obtain the chain

x(x2(x021)0(x110)2)x1
1(x2(x021)0(x110)2)x1
1(x2(x021)0(x110)2)01

of completely bracketed sequences.

We have now assigned every sequence in $P(n, k)$ to a chain. Furthermore, since bracketing occurs on subsequences of length $k + 2$ containing exactly one x, we see that every chain contains a sequence having exactly $[n/(k + 2)]$ nonsignificant symbols (or x's). Thus every chain contains an element of $P(n, k)$ of dimension e, hence we have a Dilworth partition of $P(n, k)$.

Finally, we note that we do not have diagonal symmetry in the Dilworth partition, where the diagonal map replaces a significant symbol j by $k - j$ and fixes x. However, such symmetry is not possible in general. Specifically, for $n = 3$, $k = 2$, the vertex 111 must be in a chain with some line (since the number of vertices and of lines are both 27), say the line x11. Then either of the remaining two lines through 111, say 1x1, must be assigned to a chain with one of its two remaining vertices, 101 or 121. Then by diagonal symmetry it must be assigned to both, a contradiction.

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