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CONTINUITY PROPERTIES OF MONOTONE NONLINEAR OPERATORS IN LOCALLY CONVEX SPACES

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ABSTRACT. Let X be a real locally convex Hausdorff space, X^* its dual space, and T an operator from X into 2^{X^*}. The main results of this paper are: (i) if T is D-maximal monotone and locally bounded at each point of $D(T)$, then T is upper demicontinuous; (ii) if X is a Fréchet space, T is monotone, and $D(T)$ is open in X, then T is upper demicontinuous if and only if T is upper hemicontinuous, thus generalizing a result of [3].

Let X be a real locally convex Hausdorff (topological vector) space, X^* its dual space, and 2^{X^*} the space of subsets of X^*. We write $\langle u^*, u\rangle$ in place of $u^*(u)$ for $u \in X$ and $u^* \in X^*$. Let T be an operator from X into 2^{X^*}. The effective domain of T is the set

$$D(T) = \{u \in X: T(u) \neq \varnothing\}$$

and the graph of T is the subset of $X \times X^*$ given by

$$G(T) = \{(u, u^*): u^* \in T(u)\}.$$

T is called a *monotone operator* if $\langle u^* - v^*, u - v\rangle \geqslant 0$ for each pair of elements (u, u^*) and (v, v^*) of $G(T)$. T is called a *D-maximal monotone* operator [1] if, in addition, the following condition is satisfied: if $u \in D(T)$ and $u^* \in X^*$ such that $\langle u^* - v^*, u - v\rangle \geqslant 0$ for all $(v, v^*) \in G(T)$, then $u^* \in T(u)$. The operator T is said to be *locally bounded* at u_0 if there exists a neighborhood U of u_0 such that the set

$$T(U) = \bigcup \{T(u): u \in U\}$$

is an equicontinuous subset of X^*.

The domain $D(T)$ of T is said to be *quasi-dense* [4] if for each $u \in D(T)$ there exists a dense subset M_u of X such that whenever $v \in M_u$, then $u + tv \in D(T)$ for sufficiently small $t > 0$.

Let X and Y be topological spaces; an operator T from X into 2^Y is said to be *upper semicontinuous* if, for each u_0 in X and each neighborhood V of $T(u_0)$ in Y, there exists a neighborhood U of u_0 such that $T(u) \subset V$ whenever $u \in U$.

If X is a locally convex Hausdorff space and X^* its dual, with X^* given the weak* topology, then (i) an operator which is upper semicontinuous from X

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into 2^{X^*} is said to be *upper demicontinuous* and (ii) an operator which is upper semicontinuous from each segment $S \subset X$ into 2^{X^*} is said to be *upper hemicontinuous*.

We use the symbols "$\to$" and "$\rightharpoonup$" to denote strong and weak* convergence, respectively.

THEOREM 1. *Let X be a locally convex Hausdorff space and T a D-maximal monotone operator from X into 2^{X^*} which is assumed locally bounded at each point of $D(T)$. Then T is upper demicontinuous.*

PROOF. Suppose that T is not upper demicontinuous. Then there exists a point u_0 in $D(T)$ and a weak* neighborhood W of $T(u_0)$ such that for each neighborhood U of u_0 there exist $u \in U$ and $u^* \in T(u)$ with $u^* \notin W$. Since T is locally bounded at the point u_0, we conclude the existence of a neighborhood V of u_0 such that the set $T(V)$ is an equicontinuous subset of X^*. Hence there exist a net $\{u_\alpha\}$ in V with $u_\alpha \to u_0$ and $u_\alpha^* \in T(u_\alpha)$ such that $u_\alpha^* \notin W$ for all α. Since the set $T(V)$ is relatively weak*-compact, there exists a subnet $\{u_\beta^*\}$ of $\{u_\alpha^*\}$ with $u_\beta^* \rightharpoonup u_0^* \notin T(u_0)$. By the monotonicity of T, we know that for every $(u, u^*) \in G(T)$ and every index β we have $\langle u_\beta^* - u^*, u_\beta - u\rangle \geqslant 0$. But $u_\beta \to u_0$ and $u_\beta^* \rightharpoonup u_0^*$, so that the last inequality gives

$$\langle u_0^* - u^*, u_0 - u\rangle \geqslant 0$$

for all (u, u^*) in $G(T)$. The D-maximal monotonicity of T implies that $u_0^* \in T(u_0)$, contrary to the fact that $u_0^* \notin T(u_0)$. Therefore, T is an upper demicontinuous operator.

THEOREM 2. *Let X be a locally convex Hausdorff space and T a monotone operator from X into 2^{X^*} with a quasi-dense domain $D(T)$. Suppose that T is upper hemicontinuous from X into 2^{X^*} and for each $u \in D(T)$, $T(u)$ is an equicontinuous, weak*-closed and convex subset of X^*. Then T is D-maximal monotone.*

PROOF. Let $u_0 \in D(T)$ and $u_0^* \in X^*$ such that $\langle u^* - u_0^*, u - u_0\rangle \geqslant 0$ for all $(u, u^*) \in G(T)$. It is sufficient to show that $u_0^* \in T(u_0)$. Suppose this is not the case. Consider X^* endowed with the weak* topology. Then the dual of X^* can be identified with X. Let M_{u_0} be the dense subset of X used in the definition of $D(T)$ as a quasi-dense set. Since $T(u_0)$ is an equicontinuous, weak*-closed and convex subset of X^*, there exists a $v \in M_{u_0}$ such that

$$\langle w^* - u_0^*, v\rangle < 0 \tag{1}$$

for all $w^* \in T(u_0)$. Furthermore, there exists a $t_0(v) > 0$ such that $u_t = u_0 + tv$ lies in $D(T)$ for $0 < t < t_0(v)$. Let $\{t_n\}$ be a null sequence of real numbers such that $0 < t_n < t_0(v)$ for all n. Then $u_n = u_0 + t_n v \in D(T)$, and for any $u_n^* \in T(u_n)$ we have $t_n\langle u_n^* - u_0^*, v\rangle \geqslant 0$, i.e.,

$$\langle u_n^* - u_0^*, v\rangle \geqslant 0 \tag{2}$$

for all n. By the upper hemicontinuity of T the set $\bigcup_n T(u_n)$ is relatively weak*-compact, and so there exists a subnet $\{u_\beta^*\}$ of $\{u_n^*\}$ such that $u_\beta^* \rightharpoonup u_1^* \in T(u_0)$. It then follows from (2) that

$$\langle u_1^* - u_0^*, v\rangle \geqslant 0$$

which contradicts (1). Therefore T is D-maximal monotone.

The following theorem is a corollary of Theorems 1 and 2 and generalizes Theorem 1 of **[4]**.

THEOREM 3. *Let X be a locally convex Hausdorff space, and T a monotone operator from X into 2^{X^*} with a quasi-dense domain $D(T)$. Suppose that T is locally bounded at each point of $D(T)$, and $T(u)$ is a weak*-closed convex subset of X^* for each $u \in D(T)$. Then T is upper demicontinuous if and only if it is upper hemicontinuous.*

REMARK. If X is a Fréchet space, then a monotone operator T from X into 2^{X^*} is locally bounded at each interior point of $D(T)$ (see **[2]**). A consequence of this result and of Theorem 3 is Theorem 4 which for single-valued operators T is obtained in **[3]**.

THEOREM 4. *Let X be a Fréchet space, and T a monotone operator from X into 2^{X^*} with $D(T)$ open in X. Suppose that for each $u \in D(T)$, $T(u)$ is a weak*-closed and convex subset of X^*. Then T is upper demicontinuous if and only if it is upper hemicontinuous.*

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