ON FIXED POINTS OF NONEXPANSIVE SET-VALUED MAPPINGS

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Abstract. A theorem is proved concerning the existence of fixed points of nonexpansive set-valued mappings. This generalizes a result of Dotson, Jr. [1].

This note deals with the existence of fixed points of nonexpansive set-valued mappings from a compact subset of a complete metric space into itself. This result generalizes a theorem of Dotson, Jr. [1]

Definition. Let \((X, d)\) be a complete metric space, and let \(S\) be a subset of \(X\). We denote by \(2^S\) the set of all compact subsets of \(S\). Let \(D(x, A)\) denote the ordinary distance between \(x \in X\) and \(A \in 2^S\). Let \(F = \{f_A\}_{A \in 2^S}\) be a family of functions from \([0, 1]\) into \(2^S\) with the property that for each \(A \in 2^S\), \(f_A(1) = A\). Such a family is said to be contractive if there exists a function \(\phi: (0, 1) \to (0, 1)\) such that for all \(A, B \in 2^S\) and for all \(t \in (0, 1)\) we have

\[
H(f_A(t), f_B(t)) \leq \phi(t)H(A, B)
\]

where \(H\) is the Hausdorff metric. Such a family is said to be jointly continuous if \(f_A(t) \to f_{A_0}(t_0)\) in \(2^S\) whenever \(t \to t_0\) in \([0, 1]\) and \(A \to A_0\) in \(2^S\).

Theorem. Let \(S\) be a compact subset of a complete metric space. Suppose there exists a contractive, jointly continuous family \(F\) of functions associated with \(S\). Then any nonexpansive multi-valued mapping \(T\) of \(S\) into \(2^S\) has a fixed point in \(S\).

Proof. For each \(n = 1, 2, 3, \ldots\) let \(r_n = n/(n + 1)\), and let \(T_n: S \to 2^S\) defined as \(T_nx = f_{T_n}(r_n)\) for all \(x \in S\). \(T_n\) is a well-defined map from \(S\) into \(2^S\) for each \(n\). Also, for each \(n\), and for all \(x, y \in S\), we have

\[
H(T_nx, T_ny) = H(f_{T_n}(r_n), f_{T_n}(r_n)) = H(T_nx, T_ny) \leq \phi(r_n)H(Tx, Ty) \leq \phi(r_n)d(x, y).
\]

Hence, for each \(n\), \(T_n\) is a multi-valued contraction mapping from \(S\) into \(2^S\). Then by a theorem of Nadler, Jr. [2] there exist \(x_n \in S\) such that \(x_n \in T_nx_n\). Since \(S\) is compact, there is a subsequence \(\{x_n\}\) in \(S\) of \(\{x_n\}\) converging to \(x_0\) in \(S\). Also,

\[
x_n \in T_n(x_n) = f_{T_nx_n}(r_n) \to_{n \to \infty} f_{T_nx_0}(1) = Tx_0.
\]
as $Tx_n \to Tx_0$ and $r_n \to 1$ (by joint continuity). Since $x_n \in T_n(x_n)$ for each $j$, it follows that $x_0 \in Tx_0$. Because
\[
D(x_0, Tx_0) = d(x_0, x_n) + D(x_n, Tx_0) \\
\leq d(x_0, x_n) + H(T_n x_n, Tx_0).
\]
Hence $D(x_0, Tx_0) = 0$. As $Tx_0$ is closed, $x_0 \in Tx_0$.

Remark. Dotson, Jr. [1] has proved the theorem in a Banach space setting where the nonexpansive map is single-valued. However vector space structure of the space is not needed in the proof.

References


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