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ON FIXED POINTS OF NONEXPANSIVE SET-VALUED MAPPINGS

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ABSTRACT. A theorem is proved concerning the existence of fixed points of nonexpansive set-valued mappings. This generalizes a result of Dotson, Jr. [1].

This note deals with the existence of fixed points of nonexpansive set-valued mappings from a compact subset of a complete metric space into itself. This result generalizes a theorem of Dotson, Jr. [1]

DEFINITION. Let (X, d) be a complete metric space, and let S be a subset of X. We denote by 2^S the set of all compact subsets of S. Let $D(x, A)$ denote the ordinary distance between $x \in X$ and $A \in 2^S$. Let $F = \{f_A\}_{A \in 2^S}$ be a family of functions from $[0, 1]$ into 2^S with the property that for each $A \in 2^S$, $f_A(1) = A$. Such a family is said to be contractive if there exists a function ϕ: $(0, 1) \to (0, 1)$ such that for all A and B in 2^S and for all $t \in (0, 1)$ we have

$$H(f_A(t), f_B(t)) \leqslant \phi(t) H(A, B)$$

where H is the Hausdorff metric. Such a family is said to be jointly continuous if $f_A(t) \to f_{A_0}(t_0)$ in 2^S whenever $t \to t_0$ in $[0, 1]$ and $A \to A_0$ in 2^S.

THEOREM. *Let S be a compact subset of a complete metric space. Suppose there exists a contractive, jointly continuous family F of functions associated with S. Then any nonexpansive multi-valued mapping T of S into 2^S has a fixed point in S.*

PROOF. For each $n = 1, 2, 3, \ldots$ let $r_n = n/(n+1)$, and let T_n: $S \to 2^S$ defined as $T_n x = f_{Tx}(r_n)$ for all $x \in S$. T_n is a well-defined map from S into 2^S for each n. Also, for each n, and for all $x, y \in S$, we have

$$\begin{aligned} H(T_n x, T_n y) &= H(f_{Tx}(r_n), f_{Ty}(r_n)) \\ &\leqslant \phi(r_n) H(Tx, Ty) \leqslant \phi(r_n) d(x, y). \end{aligned}$$

Hence, for each n, T_n is a multi-valued contraction mapping from S into 2^S. Then by a theorem of Nadler, Jr. [2] there exist $x_n \in S$ such that $x_n \in T_n x_n$. Since S is compact, there is a subsequence $\{x_{n_j}\}$ in S of $\{x_n\}$ converging to x_0 in S. Also,

$$x_{n_j} \in T_{n_j}(x_{n_j}) = f_{Tx_{n_j}}(r_{nj}) \to_{j \to \infty} f_{Tx_0}(1) = Tx_0,$$

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as $Tx_{n_j} \to Tx_0$ and $r_{n_j} \to 1$ (by joint continuity). Since $x_{n_j} \in T_{n_j}(x_{n_j})$ for each j, it follows that $x_0 \in Tx_0$. Because

$$\begin{aligned} D(x_0, Tx_0) &\leqslant d(x_0, x_{n_j}) + D(x_{n_j}, Tx_0) \\ &\leqslant d(x_0, x_{n_j}) + H(T_{n_j}x_{n_j}, Tx_0). \end{aligned}$$

Hence $D(x_0, Tx_0) = 0$. As Tx_0 is closed, $x_0 \in Tx_0$.

REMARK. Dotson, Jr. [1] has proved the theorem in a Banach space setting where the nonexpansive map is single-valued. However vector space structure of the space is not needed in the proof.

REFERENCES

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