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POSITIVE DERIVATIONS ON PARTIALLY ORDERED LINEAR ALGEBRA WITH AN ORDER UNIT

TAEN-YU DAI AND RALPH DeMARR

ABSTRACT. We show that the range of a positive derivation on a Dedekind σ-complete partially ordered linear algebra with an order unit is a set of generalized nilpotents. With additional assumptions on the algebra, we show that the algebra has an important property similar to a property of the algebra of upper triangular matrices.

1. Introduction and definitions. Derivations on operator algebras have been studied extensively by various authors; see [**4**]. DeMarr has shown that every Banach algebra of bounded linear operators can be ordered so that it becomes a partially ordered linear algebra with an order unit in which norm convergence and order convergence are equivalent [**3**, Theorem, p. 637]. This motivates the study of positive derivations on Dedekind σ-complete partially ordered linear algebra (dsc-pola) with an order unit.

A dsc-pola, denoted by A, is a real linear associative algebra which satisfies the following two conditions: (1) It is partially ordered so that it is a directed partially ordered linear space and $0 \leqslant xy$ whenever $0 \leqslant x, y \in A$. (2) It is Dedekind σ-complete, i.e., if $x_n \in A$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0$, then $\inf\{x_n\}$ exists. A dsc-pola has the Archimedean property: if $x, y \in A$ and $\alpha x \leqslant y$ for all positive real numbers α, then $x \leqslant 0$. In this paper, we assume A has a multiplicative identity $1 \geqslant 0$. Let $I = \{y \in A : y \geqslant 1 \text{ and } y^{-1} \geqslant 0\}$ and then define $A_1 = I - I$. It is shown in [**1**] that A_1 is an order-closed and order-convex commutative subalgebra of A which behaves much like an algebra of real-valued functions. We call A_1 the diagonal or functional part of A. For a detailed discussion and examples of A_1, see [**1**]; there we used the term polac instead of dsc-pola.

We say that $u \in A$ is an order unit if for each $x \in A$ there exists a real number δ such that $-\delta u \leqslant x \leqslant \delta u$. The fact that A has an order unit is important in this paper. Example 3.5 shows that our main Theorem 2.3 is not true without an order unit.

A derivation f on A is a linear map from A into itself such that $f(xy) = xf(y) + f(x)y$ for all x, y in A. A positive derivation has the additional property that $f(x) \geqslant 0$ whenever $x \geqslant 0$. An inner derivation is a derivation

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having the form $f(x) = xa - ax$ for all $x \in A$, where a is some fixed element in A.

Examples of order units and positive derivations are given in §3.

In **[2]**, a diagonal projection map is defined to be a positive linear map Δ from A into A_1 such that $\Delta(1) = 1$ and $0 \leqslant \Delta(x) \leqslant x$ whenever $0 \leqslant x$. This map was used to study the structure of A. For a detailed discussion and examples of Δ, see **[2]**.

A generalized nilpotent is an element z in A such that for each $\alpha > 0$ there exists $v \in A$, depending on α, satisfying the inequality $-v \leqslant \alpha^n z^n \leqslant v$ for all positive integers n. Note that every nilpotent in A is a generalized nilpotent.

LEMMA 1.1. *If $y, z \in A$ and $-z \leqslant y \leqslant z$, then $-z^n \leqslant y^n \leqslant z^n$ for all positive integers n. Hence, if z is a generalized nilpotent, then y is also a generalized nilpotent. In particular, if z is a nilpotent, then y too is a nilpotent.*

The proof is left as an exercise for the reader.

2. Main results. We assume that A has an order unit and that f is a positive linear map from A into itself which satisfies the inequality $f(xy) \geqslant xf(y) + f(x)y$ whenever $x \geqslant 0$ and $y \geqslant 0$. This not only describes a positive derivation, but it also enables us to study a certain kind of homomorphism which is discussed in Example 3.3. Although an order unit is not unique, in this paper it is desirable to choose an order unit u such that $u \leqslant u^2 \leqslant \beta u$, where $\beta \geqslant 1$ and β is close to 1. The range of f is described in the following theorems.

THEOREM 2.1. *If $u \in A$ is an order unit such that $u \leqslant u^2 \leqslant \beta u$, where $1 \leqslant \beta < 2$, then $f(x) = 0$ for all $x \in A$.*

PROOF. Since u is an order unit and f is a positive linear map, we need only show that $f(u) = 0$. Let $z = f(u)$. First applying f to both sides of the inequality $u^2 \leqslant \beta u$, then using the definition of f, we obtain $uz + zu \leqslant f(u^2) \leqslant \beta z$. Thus, $2uzu \leqslant u^2zu + uzu^2 \leqslant \beta uzu$. This implies that $(\beta - 2)uzu \geqslant 0$. Since $\beta < 2$, it follows that $0 \geqslant uzu \geqslant 0$ or $uzu = 0$. Since u is an order unit, we can find some number δ such that $\delta u \geqslant 1$. Now clearly we have $0 = (\delta u)z(\delta u) \geqslant z \geqslant 0$. Therefore, $z = 0$. □

THEOREM 2.2. *If $u \in A$ is an order unit such that $u \leqslant u^2 \leqslant 2u$, then $f(x)^2 = 0$ for all $x \in A$.*

PROOF. Let $z = f(u)$. We first show that $z^2 = 0$. As in Theorem 2.1, we obtain $2uzu \leqslant u^2zu + uzu^2 \leqslant 2uzu$. Thus, $2uzu = u^2zu + uzu^2$ or $0 \leqslant (u^2 - u)zu = uz(u - u^2) \leqslant 0$. Hence, $(u^2 - u)zu = 0$. Since $(u^2 - u)z \geqslant 0$ and $\delta u \geqslant 1$ for some $\delta > 0$, we get $0 = (u^2 - u)z(\delta u) \geqslant (u^2 - u)z \geqslant 0$ or $(u^2 - u)z = 0$. It follows that $0 = f((u^2 - u)z) \geqslant (u^2 - u)f(z) + f(u^2 - u)z \geqslant 0$. Since both terms $f(u^2 - u)z$ and $(u^2 - u)f(z)$ are nonnegative, we have

$0 = f(u^2 - u)z = (f(u^2) - z)z = f(u^2)z - z^2 \geqslant (uz + zu)z - z^2$. Hence, $uz^2 + zuz \leqslant z^2$. Multiplying by u on both sides of the last inequality, we obtain $u^2z^2 + (uz)^2 \leqslant uz^2 \leqslant u^2z^2$. Consequently, $(uz)^2 = 0$. From $\delta u \geqslant 1$ we know that $\delta uz \geqslant z \geqslant 0$, so that $z^2 = 0$. Now for any $x \in A$ we have $-\alpha u \leqslant x \leqslant \alpha u$ for some α. Thus, $-\alpha z \leqslant f(x) \leqslant \alpha z$. Using Lemma 1.1 and the fact that $z^2 = 0$, we conclude that $f(x)^2 = 0$. □

THEOREM 2.3. *If $u \in A$ is an order unit such that $u \leqslant u^2 \leqslant \beta u$, where $\beta > 2$, then $f(x)$ is a generalized nilpotent for all $x \in A$.*

PROOF. Again, we need only show that $z = f(u)$ is a positive generalized nilpotent. From $uz + zu \leqslant f(u^2) \leqslant \beta z$ we see that $f(u)z + zf(u) \leqslant f(uz + zu) \leqslant \beta f(z)$ or $2z^2 \leqslant \beta f(z)$. By induction we will show that $2nz^{n+1} \leqslant \beta f(z^n)$ for every positive integer n. We have proved the inequality is true for $n = 1$. Assume the inequality is true when $n = k$. Hence, $\beta f(z^{k+1}) \geqslant \beta f(z^k)z + \beta z^k f(z) \geqslant 2kz^{k+1}z + 2z^kz^2 = 2(k+1)z^{k+2}$. Thus, the induction is complete.

Let $1 \leqslant \lambda$ be a number such that $z \leqslant \lambda u$. Next, using induction again we will show that $n!z^n \leqslant (\beta\lambda)^n u$ for every positive integer n. Clearly the inequality is true for $n = 1$. Suppose the inequality holds for $n = k$. Hence, $k!\beta f(z^k) \leqslant \beta(\beta\lambda)^k f(u) \leqslant \beta(\beta\lambda)^k \lambda u = (\beta\lambda)^{k+1}u$. From the previous paragraph we have $\beta f(z^k) \geqslant 2kz^{k+1} \geqslant (k+1)z^{k+1}$; therefore, $(\beta\lambda)^{k+1}u \geqslant k!(k+1)z^{k+1} = (k+1)!z^{k+1}$. Thus, the second induction is complete. Now for any number $\alpha > 0$ we see that $n!\alpha^n z^n \leqslant (\alpha\beta\lambda)^n u$ or $0 \leqslant \alpha^n z^n \leqslant (\alpha\beta\lambda)^n u/n!$ for all positive integers n. Since $\{(\alpha\beta\lambda)^n/n!\}$ is a bounded sequence, there exists a real number γ such that $0 \leqslant \alpha^n z^n \leqslant \gamma u$ for all n (of course, γ depends on α). This shows that z is a positive generalized nilpotent. Since u is an order unit and f is a positive linear map, for any $x \in A$ there exists a number δ such that $-\delta u \leqslant x \leqslant \delta u$ and therefore $-\delta z \leqslant f(x) \leqslant \delta z$. Using Lemma 1.1 and the fact that z is a positive generalized nilpotent, we may conclude that $f(x)$ is a generalized nilpotent. □

Conditions were given in [**2**] which imply that a diagonal projection map Δ is multiplicative; that is, $\Delta(xy) = \Delta(x)\Delta(y)$ for all x, y in A. Roughly speaking, if A has a diagonal projection map Δ which is multiplicative, then A is like an algebra of upper triangular matrices. The examples in the following section will illustrate this. The reader should observe the following interesting points. In Example 3.1, the matrix $1 + z$ $(z = f(u))$ is not an order unit of A; in fact $1 + z = 1$. Note that Δ is not multiplicative. However, in Examples 3.2, 3.3, and 3.4, the matrix $1 + z$ is an order unit. Note that Δ is multiplicative in each case. These observations lead to the following theorem.

THEOREM 2.4. *If a diagonal projection map Δ exists on A and if $1 + z$ $(z = f(u)$, u as in Theorem 2.3) is an order unit, then Δ is multiplicative.*

PROOF. We first show that $\Delta(z) = \Delta(z^2) = 0$. Since z is a positive generalized nilpotent, for each $\alpha > 0$ there exists $v \in A$ such that $0 \leqslant \alpha^n z^n \leqslant v$ for all positive integers n. Using Lemma 3.4 of [2], we have $0 \leqslant \Delta(\alpha^n z^n) = \alpha^n \Delta(z^n) \leqslant 1$ for all n. Since this inequality holds for arbitrary $\alpha > 0$ and since A has the Archimedean property, we conclude that $\Delta(z^n) = 0$ for all n; in particular, $\Delta(z) = \Delta(z^2) = 0$.

By Lemma 4.1 of [2], to prove that Δ is multiplicative we need only verify that if $0 \leqslant w \in A$ and $\Delta(w) = 0$, then $\Delta(w^2) = 0$.

Since $1 + z$ is an order unit, we can find some $\delta > 0$ such that $0 \leqslant w \leqslant \delta(1 + z)$. By the fact that $\Delta(z) = 0$ and by the definition of Δ, we see that $0 \leqslant \delta(1 + z) - w - \Delta(\delta(1 + z) - w) = \delta z - w$. Hence, $0 \leqslant w^2 \leqslant \delta^2 z^2$. Let us apply Δ throughout the last inequality. The fact that Δ is a positive linear map gives $0 \leqslant \Delta(w^2) \leqslant \delta^2 \Delta(z^2) = 0$. Therefore, $\Delta(w^2) = 0$ which means that Δ is multiplicative. □

3. Examples.

EXAMPLE 3.1. Let A be the matrix algebra of all m-by-m real matrices with componentwise ordering. Hence, A is a dsc-pola. If u is the matrix with all entries equal to $1/m$, then u is an order unit. Since $u = u^2$, we have $\beta = 1$. By Theorem 2.1 we conclude that the only positive derivation on A is the zero derivation. It is easy to check that in this example A_1 is the set of all diagonal matrices in A. For $x \in A$ define $\Delta(x)$ to be the diagonal part of x. We can easily check that Δ is a diagonal projection map which is not multiplicative.

EXAMPLE 3.2. Let A be the matrix algebra of all m-by-m real upper triangular matrices with componentwise ordering. Hence, A is a dsc-pola. The matrix u in A with diagonal entries equal to 1 and $\varepsilon > 0$ elsewhere is an order unit. Note that $u \leqslant u^2 \leqslant \beta u$, where $\beta = 2 + (m - 2)\varepsilon$. If $m = 2$, then $\beta = 2$. Take the diagonal matrix $a = [i\delta_{ij}]$, where δ_{ij} is the Kronecker delta and define $f(x) = xa - ax$ for $x \in A$. It is easy to check that f is a positive inner derivation and that $f(x)^m = 0$ for all $x \in A$. The set A_1 is the collection of all diagonal matrices in A. If we define Δ as in Example 3.1, then Δ is a multiplicative diagonal projection map.

EXAMPLE 3.3. Let A be the dsc-pola described in Example 3.2. Define $g(x) = a^{-1}xa$, where $x \in A$ and a is the same diagonal matrix as in Example 3.2. We can verify easily that g is linear, $g(xy) = g(x)g(y)$ for all $x, y \in A$ and $g(x) \geqslant x \geqslant 0$ whenever $x \geqslant 0$. Let f be a linear map on A defined by $f(x) = g(x) - x$. Note that for $0 \leqslant x, y$ we have $f(x) \geqslant 0, f(y) \geqslant 0$ and

$$\begin{aligned} f(xy) &= g(xy) - xy = g(x)g(y) - xy \\ &= (f(x) + x)(f(y) + y) - xy \geqslant xf(y) + f(x)y. \end{aligned}$$

This is an example in which the inequality can be strict. Furthermore, $f(x)^m = 0$ for all $x \in A$.

In general, if $g(x)$ is a homomorphism of a dsc-pola A into itself such that $g(x) \geqslant x$ for $x \geqslant 0$, then $f(x) = g(x) - x$ satisfies the conditions at the beginning of §2. Theorem 2.3 shows that if A has an order unit, then the difference $g(x) - x$ is a generalized nilpotent for all $x \in A$.

EXAMPLE 3.4. Let A be the algebra of matrices of the form $x = \left[\begin{smallmatrix} \alpha & \gamma \\ 0 & \alpha \end{smallmatrix}\right]$, where α and γ are reals. If A is partially ordered componentwise, then A is a dsc-pola with $u = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ as an order unit and $u \leqslant u^2 \leqslant 2u$. Since A is commutative, there is no inner derivation on A, but the map $f(x) = \left[\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right]$ is a positive derivation. Note that $f(x)^2 = 0$. The set A_1 is the collection of all scalar matrices in A. If we define Δ as in Example 3.1, then Δ is a multiplicative diagonal projection map.

EXAMPLE 3.5. Let A be the algebra of all real polynomials of the form $x(\tau) = \alpha_0 + \alpha_1\tau + \alpha_2\tau^2 + \cdots + \alpha_n\tau^n$. The ordering $0 \leqslant x$ means that the coefficients of $x(\tau)$ are nonnegative. A is a dsc-pola with no order unit. The ordinary derivative f is a positive derivation. Let $y(\tau) = \tau$. We see that $f(y) = 1$ is not a generalized nilpotent. This example shows that Theorem 2.3 is not true if A is an arbitrary dsc-pola.

EXAMPLE 3.6. We now give an example in which $z = f(u)$ is a generalized nilpotent and not just a nilpotent. Let B be the algebra of all infinite matrices $x = [\alpha_{ij}]$, where $i, j = 1, 2, 3, \ldots$, which are in upper triangular form. This means that $\alpha_{ij} = 0$ for $i > j$. If we partially order B componentwise, then B is a dsc-pola. Of course, B does not have an order unit, but we may construct subalgebras of B which do have order units. We do this by first selecting any matrix s in B such that $0 \leqslant s^2 \leqslant s$ and then we define $u = 1 + s$. It is easy to show that $u \leqslant u^2 \leqslant 3u$. We next define

$$A = \bigcup_{k=1}^{\infty} \{x\colon -ku \leqslant x \leqslant ku\}.$$

Of course, A is an order-convex subalgebra of B which has u as an order unit. Since A is order-convex, it is also a dsc-pola. The reader should note that A depends on u.

Let us now define the fixed diagonal matrix $a = [\alpha_i \ \delta_{ij}] \in B$, where $0 \leqslant \alpha_1 < \alpha_2 < \cdots \leqslant 1$, and then define the positive inner derivation $f(x) = xa - ax$ for all $x \in B$. Since $0 \leqslant a \leqslant 1$, it follows that $f(x) \leqslant x$ whenever $0 \leqslant x$. Hence, we may regard $f(x)$ as a derivation which maps A into itself.

In particular, let us select $s = [\sigma_{ij}] \in B$, where $\sigma_{ij} = 2^{-j}$ whenever $1 \leqslant i \leqslant j$. It is easy to verify that $0 \leqslant s^2 \leqslant s$. Note that $f(s)$ has a positive off-diagonal entry in any position where s has a positive off-diagonal entry. Using this fact and the fact that $z = f(u) = f(s)$, we see that z is a generalized nilpotent, but z^n is not zero for any positive integer n.

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Department of Mathematics, York College (CUNY), Jamaica, New York 11451

Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131