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ON MARKOV STABILITY

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ABSTRACT. The concept of T-stability for vector-valued functions is introduced–a generalization of strong stability in the sense of Markov. Moreover, for solutions of T-periodic systems of differential equations, T-stability is compared with Liapunov stability and it is shown that boundedness and T-stability imply asymptotic almost periodicity.

1. Introduction. Markov defined strong stability in the following way: the function x: $\mathbf{R} \to \mathbf{R}^n$ is called *strongly stable* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if α and β are any two real numbers with $\|x(\alpha) - x(\beta)\| < \delta$, then $\|x(t + \alpha) - x(t + \beta)\| < \varepsilon$ for all $t \in \mathbf{R}$ [**1**] (by the way, he also defined strong stability in the past and strong stability in the future). The importance of this concept results from the Markov theorem: if the function $x(t)$ is continuous, bounded and strongly stable, then $x(t)$ is almost periodic [**1**] (see also [**2**, p. 181]).

In the present paper we define a weaker type of stability, requiring that Markov condition holds only for α and β integral multiples of some real number $T > 0$ (independent of ε), and distinguish between *T-stability in the past and in the future* (see Definition 1 of §2) and *T-stability in the future* only (shortly *T-stability*) (see Definition 2 of §2). Moreover, we establish the following theorems (Theorems 1 and 2): if the function $x(t)$ is uniformly continuous and bounded, then T-stability in the past and in the future (T-stability) implies that $x(t)$ is almost periodic (asymptotically almost periodic). Theorem 1 is not a generalization of the Markov theorem, because we assume the *uniform* continuity of $x(t)$. As a matter of fact Theorems 1 and 2 become interesting especially in the theory of periodic systems of differential equations, because boundedness of a solution $x(t)$ implies the uniform continuity of $x(t)$. Our main result is (Theorem 3): if a solution $x(t)$ of a T-periodic system is bounded in the future and T-stable, then $x(t)$ is asymptotically almost periodic. At last we show that T-stability of a solution $x(t)$ of a T-periodic system of differential equations is nothing else than *conditional stability* of $x(t)$ with respect to the solutions $x(t + mT)$, $m = 1, 2, \ldots$. Therefore T-stability is weaker than Liapunov stability, and by Theorem 3 we obtain the Corollary of §3 which shows that the condition of *uniform* Liapunov stability, usually adopted in the literature to assure

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asymptotic almost periodicity, is unnecessarily strong.

2. Definition of T-stability. Let $x(t)$ be a vector-valued function of the real variable t, defined on the real line $\mathbf{R}$ with values in $\mathbf{R}^n$ (the Euclidean space of dimension n) and let $\|\cdot\|$ designate any norm on $\mathbf{R}^n$. We give the following

DEFINITION 1. The function $x(t)$ is called *T-stable in the past and in the future* if there exists a $T > 0$ such that

$$(\forall \varepsilon > 0)(\exists \delta > 0)\big(\forall n, m \in \mathbf{Z}: \|x(nT) - x(mT)\| < \delta\big)(\forall t \in \mathbf{R})$$
$$\|x(t + nT) - x(t + mT)\| < \varepsilon, \tag{2.1}$$

where $\mathbf{Z}$ is the set of relative integers.

The following theorem holds

THEOREM 1. *If the function x: $\mathbf{R} \to \mathbf{R}^n$ satisfies the following conditions*

(i) *$x(t)$ is uniformly continuous and bounded on* $\mathbf{R}$,

(ii) *$x(t)$ is T-stable in the past and in the future* (*according to Definition* 1), *then $x(t)$ is almost periodic.*

Let $\{\alpha_n\}$ be a sequence of real numbers. Then for every n ($n = 1, 2, \ldots$) there is $k_n \in \mathbf{Z}$ such that $k_nT \leqslant \alpha_n < (k_n + 1)T$. Put $\beta_n = \alpha_n - k_nT$. Owing to the boundedness of the sequences $\{\beta_n\}$ and $\{x(k_nT)\}$, we can extract two common subsequences of $\{\beta_n\}$ and $\{k_n\}$, respectively denoted by $\{\beta_n'\}$ and $\{k_n'\}$, such that the sequences $\{\beta_n'\}$ and $\{x(k_n'T)\}$ are convergent. Let us consider the subsequence $\{\alpha_n'\}$ of $\{\alpha_n\}$, where $\alpha_n' = \beta_n' + k_n'T$. We prove now that the sequence of functions $\{x(t + \alpha_n')\}$ is uniformly convergent on $\mathbf{R}$, i.e. $x(t)$ is almost periodic like Bochner. In fact, for a given $\varepsilon > 0$, there correspond $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that the conditions of uniform continuity and T-stability in the past and in the future of $x(t)$ are respectively satisfied. Furthermore there exists a positive integer $N = N(\varepsilon)$ such that for all $n, m \geqslant N$ it is: $|\beta_n' - \beta_m'| < \delta_1$, $\|x(k_n'T) - x(k_m'T)\| < \delta_2$. Then for all $n, m \geqslant N$ and all $t \in \mathbf{R}$ we have

$$\begin{aligned}\|x(t + \alpha_n') - x(t + \alpha_m')\| \leqslant{}& \|x(t + \beta_n' + k_n'T) - x(t + \beta_N' + k_n'T)\| \\ &+ \|x(t + \beta_N' + k_n'T) - x(t + \beta_N' + k_m'T)\| \\ &+ \|x(t + \beta_N' + k_m'T) - x(t + \beta_m' + k_m'T)\| < 3\varepsilon,\end{aligned}$$

which proves the theorem.

Let $x(t)$ be a vector-valued function defined on some interval $[t_0, \infty)$. We introduce the concept of *T-stability in the future*, shortly denoted by *T-stability*, as follows

DEFINITION 2. The function x: $[t_0, \infty) \to \mathbf{R}^n$ is called *T-stable* if there exists a $T > 0$ such that

$$(\forall \varepsilon > 0)(\exists \delta > 0)\big(\forall n, m \in \mathbf{N}: \|x(t_0 + nT) - x(t_0 + mT)\| < \delta\big)(\forall t \geqslant t_0)$$
$$\|x(t + nT) - x(t + mT)\| < \varepsilon, \tag{2.2}$$

where $\mathbf{N}$ is the set of positive integers.

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The following theorem holds

THEOREM 2. *If the function* $x\colon [t_0, \infty) \to \mathbf{R}^n$ *is*
(i) *uniformly continuous and bounded on* $[t_0, \infty)$,
(ii) *T-stable* (*according to Definition* 2),
then $x(t)$ *is asymptotically almost periodic.*

The concept of asymptotic almost periodicity was introduced by Fréchet [3], who also gave the equivalent condition: the continuous function $x\colon [t_0, \infty) \to \mathbf{R}^n$ is asymptotically almost periodic if and only if for every sequence $\{\alpha_n\}$, with $\alpha_n \to \infty$ for $n \to \infty$, there is a subsequence $\{\alpha_n'\}$ such that $\{x(t + \alpha_n')\}$ is uniformly convergent on $[t_0, \infty)$. Using this condition, the proof of Theorem 2 is similar to that of Theorem 1 and is therefore omitted.

3. Periodic systems: comparison between Liapunov stability and *T*-stability. Let us consider a periodic system of n differential equations

$$\dot{x} = f(t, x) \tag{3.1}$$

where $x, f \in \mathbf{R}^n$ and the function $f(t, x)$ is defined and continuous on $\mathbf{R} \times \Omega$, where Ω is a connected open set of $\mathbf{R}^n$, and $f(t + T, x) \equiv f(t, x)$, for some period $T > 0$. Moreover, let the right-hand side of (3.1) be smooth enough to ensure the uniqueness of the solutions. Then the maximal solution of (3.1) through $(t_0, x_0) \in \mathbf{R} \times \Omega$ will be denoted by $\varphi(t, t_0, x_0)$ and by definition it is $\varphi(t_0, t_0, x_0) = x_0$.

An equivalent condition of T-stability for a solution $x(t)$ of (3.1) defined for $t \geqslant t_0$, is given by the following

LEMMA. *A necessary and sufficient condition for a solution* $x\colon [t_0, \infty) \to \mathbf{R}^n$ *of the T-periodic system* (3.1) *to be T-stable is*

$$(\forall \varepsilon > 0)(\exists \delta > 0)\big(\forall m \in \mathbf{N}\colon \|x(t_0) - x(t_0 + mT)\| < \delta\big)(\forall t \geqslant t_0)$$
$$\|x(t) - x(t + mT)\| < \varepsilon. \tag{3.2}$$

Condition (3.2) is obtained from (2.2), putting $n = 0$. Let us show that condition (3.2) implies T-stability with the same pair $(\varepsilon, \delta(\varepsilon))$. Let $\bar{n}, \bar{m} \in \mathbf{N}$ ($\bar{n} \geqslant \bar{m}$) be given such that $\|x(t_0 + \bar{n}T) - x(t_0 + \bar{m}T)\| < \delta$. With the change of variable: $t \to t + \bar{m}T$ and for the particular value $m = \bar{n} - \bar{m} \geqslant 0$, we obtain from (3.2) the following relation: $\|x(t_0 + \bar{m}T) - x(t_0 + \bar{n}T)\| < \delta \Rightarrow \forall t \geqslant t_0$, $\|x(t + \bar{m}T) - x(t + \bar{n}T)\| < \varepsilon$, which proves the lemma.

Let us suppose that the system (3.1) possesses a solution $x(t)$ *bounded in the future*, i.e. defined for $t \geqslant t_0$ and such that $x(t) \in K \subset \Omega$ for $t \geqslant t_0$, where K is a compact set of $\mathbf{R}^n$. Then $x(t)$ is uniformly continuous on $[t_0, \infty)$, due to the continuity and periodicity of the right-hand side of (3.1), and therefore taking into account the lemma, Theorem 2 of §2 can be reformulated in the following way

THEOREM 3. *Let* $x(t)$ *be a solution of the T-periodic system* (3.1) *such that*

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(i) *$x(t)$ is bounded in the future,*
(ii)

$$(\forall \varepsilon > 0)(\exists \delta > 0)\big(\forall m \in \mathbf{N}: \|x(t_0) - x(t_0 + mT)\| < \delta\big)(\forall t \geqslant t_0)$$
$$\|x(t) - x(t + mT)\| < \varepsilon.$$

Then $x(t)$ is asymptotically almost periodic.

Let us now compare T-stability with Liapunov stability, which is defined as follows: the solution $x: [t_0, \infty) \to \mathbf{R}^n$ is *Liapunov stable* if, for every $\varepsilon > 0$ and every $t_0' \geqslant t_0$ there is a $\delta = \delta(\varepsilon, t_0') > 0$ such that for any x_0' with $\|x_0' - x(t_0')\| < \delta$ there follows $\|x(t) - \varphi(t, t_0', x_0')\| < \varepsilon$ for all $t \geqslant t_0'$. Then taking $t_0' = t_0$ and $x_0' = x(t_0 + mT)$, $m = 1, 2, \ldots,$ we obtain condition (3.2), which is equivalent to T-stability (remark that $\delta = \delta(\varepsilon, t_0)$ depends only on ε, because t_0 has to be considered as *fixed*). This shows that for periodic systems T-stability is weaker than Liapunov stability. In fact, T-stability is a *conditional stability*, i.e. the solution $x(t)$ is stable at t_0 *only* with respect to the particular solutions corresponding to the initial conditions $(t_0, x(t_0 + mT))$, i.e. the solutions $x(t + mT)$, $m = 1, 2, \ldots$. Therefore the following corollary of Theorem 3 holds

COROLLARY. *If a solution $x(t)$ of the periodic system* (3.1) *is bounded in the future and Liapunov stable, then $x(t)$ is asymptotically almost periodic.*

This result shows that the condition of *uniform* stability in the sense of Liapunov (which is always assumed in the literature to assure the asymptotic almost periodicity of a solution bounded in the future, beginning with A. Halanay [**4**, p. 486]) is unnecessarily strong. In fact, for a *nontrivial* solution of a periodic system, stability and uniform stability are not equivalent (see [**5**, p. 42]).

REMARK. The results obtained in the present section can be extended to periodic systems of differential equations for which the uniqueness of solutions does not hold.

REFERENCES

1. A. Markov, *Stabilität im Liapounoffschen Sinne und Fastperiodizität*, Math. Z. **36** (1933), 708–738.

2. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Math., vol. 377, Springer-Verlag, New York, 1974.

3. M. Fréchet, *Les fonctions asymptotiquement presque-périodiques*, Rev. Sci. **79** (1941), 341–354.

4. A. Halanay, *Differential equations: Stability, oscillations, time lags*, Academic Press, New York, 1966.

5. T. Yoshizawa, *Stability theory and the existence of periodic solutions and almost periodic solutions*, Appl. Math. Sci. 14, Springer-Verlag, New York, 1975.

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