PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

THE SIZE OF THE SET OF LEFT INVARIANT MEANS ON AN ELA SEMIGROUP

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ABSTRACT. Let S be an ELA semigroup and let $\mathrm{m}(S)$ be the smallest possible cardinality of the set $\{s \in S\colon Fs = \{s\}\}$ as F ranges over the finite subsets of S. The main purpose of this note is to show that if $\mathrm{m}(S)$ is infinite, then S has exactly $2^{2^{\mathrm{m}(S)}}$ (multiplicative) left invariant means.

A semigroup S is said to be extremely left amenable (ELA) if there exists a multiplicative left invariant mean on S. (See [7], [3], [4], [5].) We denote by $\mathfrak{L}_m(S)$ ($\mathfrak{L}(S)$) the set of multiplicative left invariant means (left invariant means) on S. The cardinality of a set X is denoted by $|X|$.

Investigations into the sizes of the sets of left invariant means on an infinite, left amenable, cancellative semigroup and on an amenable locally compact group are made in [1], [2], [8], [9], [10] and [6]. In particular, Chou shows in [2] that the cardinality of the set of left invariant means on an infinite, discrete, amenable group is $2^{2^{|G|}}$. Our aim is to prove a limilar result for the ELA semigroup S.

The family of finite subsets of a set X is denoted by $\mathfrak{F}(X)$. If $F \in \mathfrak{F}(S)$, we set

$$Z_F = \{s \in S\colon Fs = \{s\}\}.$$

By [3, p. 185], Z_F is never empty. It follows that if

$$\mathrm{m}(S) = \min\{|Z_F|\colon F \in \mathfrak{F}(S)\}$$

then $\mathrm{m}(S) > 0$.

THEOREM. *Let S be an ELA semigroup. If $\mathrm{m}(S)$ is infinite, then*

$$|\mathfrak{L}_m(S)| = |\mathfrak{L}(S)| = 2^{2^{\mathrm{m}(S)}}.$$

If $\mathrm{m}(S)$ is finite, then $|\mathfrak{L}_m(S)| = \mathrm{m}(S)$, and $\mathfrak{L}(S)$ is the convex hull in $l_\infty(S)^*$ of $\mathfrak{L}_m(S)$ (and so has cardinality 1 if $\mathrm{m}(S) = 1$, and cardinality $\mathfrak{c}$ if $1 < \mathrm{m}(S) < \infty$).

PROOF. Let $F_0 \in \mathfrak{F}(S)$ be such that $|Z_{F_0}| = \mathrm{m}(S)$. Let $Z = Z_{F_0}$.

Suppose firstly that $\mathrm{m}(S)$ is finite. It follows from [5, Lemma 1] and from [4, Theorem A] that $Z = \mathfrak{L}_m(S)$ and $\mathfrak{L}(S) = \mathrm{co}\, Z$ in $l_\infty(S)^*$.

Now suppose that $\mathrm{m}(S)$ is infinite, and let α be the smallest ordinal of cardinality $\mathrm{m}(S)$. We construct by transfinite recursion, a disjoint family $\{\theta_\varepsilon$:

Received by the editors September 12, 1977.
AMS (MOS) subject classifications (1970). Primary 43A07.

$\varepsilon \in \alpha\}$ of subsets of Z such that $\theta_\varepsilon \cap Z_F \neq \varnothing$ for all $\varepsilon \in \alpha$ and all $F \in \mathfrak{F}(S)$.

To this end, we well-order $\mathfrak{F}(Z) = \{F_\beta: \beta \in \alpha\}$, and construct for each $\beta \in \alpha$, a family $\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta\}$ of subsets of Z such that the following conditions are satisfied:

(i) $\Delta_\varepsilon^\beta \cap \Delta_\eta^\beta = \varnothing$ if $\varepsilon \neq \eta$;
(ii) $|\Delta_\varepsilon^\beta| \leqslant |\beta|$ if β is infinite, and Δ_ε^β is finite if β is finite;
(iii) $Z_{F_\delta} \cap \Delta_\varepsilon^\beta \neq \varnothing$ whenever $\varepsilon \leqslant \delta \leqslant \beta$;
(iv) $\Delta_\varepsilon^\beta \subset \Delta_\varepsilon^{\beta_1}$ whenever $\varepsilon \leqslant \beta \leqslant \beta_1$.

Suppose that $\gamma \in \alpha$ and that the families $\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta\}$, satisfying conditions (i)–(iv), have been constructed for $\beta < \gamma$. From (ii), we have

$$\left|\bigcup\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta < \gamma\}\right| \leqslant \mathrm{m}(S).$$

Let $\{x_\varepsilon^\gamma: \varepsilon \leqslant \gamma\}$ be a set of distinct points in $Z_{F_\gamma} \sim [\bigcup\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta < \gamma\}]$ and define $\Delta_\varepsilon^\gamma = [\bigcup\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta < \gamma\}] \cup \{x_\varepsilon^\gamma\}$ $(\varepsilon < \gamma)$, $\Delta_\gamma^\gamma = \{x_\gamma^\gamma\}$. Then the families $\{\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta\}: \beta \leqslant \gamma\}$ satisfy the conditions (i)–(iv).

This completes the construction of the families $\{\Delta_\varepsilon^\beta: \varepsilon \leqslant \beta\}$ for all $\beta \in \alpha$. Now set $\theta_\varepsilon = \bigcup\{\Delta_\varepsilon^\beta: \beta \in \alpha\}$ for all $\varepsilon \in \alpha$. The family $\{\theta_\varepsilon: \varepsilon \in \alpha\}$ is disjoint. Let $F \in \mathfrak{F}(S)$ and $\varepsilon \in \alpha$. We check that $\theta_\varepsilon \cap Z_F \neq \varnothing$. Let $x \in Z_{F_0} \cap Z_F$. Since $Z_{\{x\}} \subset Z_F$, it suffices to show that $\theta_\varepsilon \cap Z_{\{x\}} \neq \varnothing$. Suppose not. Then $\theta_\varepsilon \cap Z_G = \varnothing$ for all $G \in \mathfrak{F}(Z)$ containing x. But the set of such G is of cardinality $\mathrm{m}(S)$, whereas, using (iii), the set of such G has cardinality at most $|\varepsilon| < \mathrm{m}(S)$. So $\theta_\varepsilon \cap Z_F \neq \varnothing$.

We now proceed as in [**10**, p. 158]. From [**2**, Lemma 2], there exists a collection $\{P_\delta: \delta \in \Delta\}$ of subsets of α where $|\Delta| = 2^{\mathrm{m}(S)}$ and $\bigcap_{i=1}^n P_{\delta_i}^{\varepsilon(i)} \neq \varnothing$ whenever $\delta_1, \ldots, \delta_n$ are distinct points of Δ and $\varepsilon(i) = 1$ or c. (Note that c denotes complementation.) For each $\delta \in \Delta$, set $X_\delta = \bigcup\{\theta_\varepsilon: \varepsilon \in P_\delta\}$. Let $\delta_1, \ldots, \delta_n, \zeta_1, \ldots, \zeta_m$ be distinct points of Δ and let $x_1, \ldots, x_n, y_1, \ldots, y_m \in S^1$, where S^1 is the semigroup S with identity adjoined. Then there exists γ such that

$$\theta_\gamma \subset \left(\bigcap_{i=1}^n X_{\delta_i}\right) \cap \left(\bigcap_{i=1}^m X_{\zeta_i}^c\right).$$

Then if $s \in \theta_\gamma \cap Z_{\{x_1, \ldots, x_n, y_1, \ldots, y_m\}}$, we have

$$s \Cup \left(\bigcap_{i=1}^n x_i X_{\delta_i}\right) \cap \left(\bigcap_{i=1}^m y_i X_{\zeta_i}^c\right).$$

It follows that if $\varepsilon: \Delta \to \{1, c\}$, then there is a point of $\mathfrak{L}_m(S)$ in the set $\bigcap\{(\widehat{xX_\delta^{\varepsilon(\delta)}}): x \in S^1, \delta \in \Delta\}$, where $\hat{A}$ is the closure in βS of a subset A of S. So $|\mathfrak{L}_m(S)| \geqslant 2^{2^{\mathrm{m}(S)}}$. Using [**5**, Lemma 1] and [**4**, Theorem A], we have $\mathfrak{L}_m(S) \subset \beta Z$ and $\mathfrak{L}(S) \subset l_\infty(Z)^*$. Since $|\beta Z| = 2^{2^{\mathrm{m}(S)}} = |l_\infty(Z)^*|$, the desired result follows. □

References

1. C. Chou, *On the size of the set of left invariant means on a semigroup*, Proc. Amer. Math. Soc. **23** (1969), 199–205.

2. ______, *The exact cardinality of the set of invariant means on a group*, Proc. Amer. Math. Soc. **55** (1976), 103–106.

3. E. Granirer, *Extremely amenable semigroups*, Math. Scand. **17** (1965), 177–197.

4. ______, *Extremely amenable semigroups*. II, Math. Scand. **20** (1967), 93–113.

5. ______, *Functional analytic properties of extremely amenable semigroups*, Trans. Amer. Math. Soc. **137** (1969), 53–75.

6. M. Klawe, *On the dimension of left invariant means and left thick subsets*, Trans. Amer. Math. Soc. **231** (1977), 507–518.

7. T. Mitchell, *Fixed points and multiplicative left invariant means*, Trans. Amer. Math. Soc. **122** (1966), 195–207.

8. J. M. Rosenblatt, *Invariant means and invariant ideals in $L_\infty(G)$ for a locally compact group G*, J. Functional Analysis **21** (1976), 31–51.

9. ______, *Invariant means for the bounded measurable functions on a locally compact group*, Math. Ann. **220** (1976), 219–228.

10. ______, *The number of extensions of an invariant mean*, Compositio Math. **33** (1976), 147–159.

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