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ON TOPOLOGICAL ISOMORPHISMS OF $C_0(X)$ AND THE CARDINAL NUMBER OF X

BAHATTIN CENGIZ

ABSTRACT. In this paper it is proved that if $C_0(X)$ and $C_0(Y)$ are topologically isomorphic then $|X| = |Y|$.

If X is a locally compact space,[1] we denote by $C_0(X)$ the Banach space of all continuous, complex-valued functions vanishing at infinity on X, provided with the usual supremum norm. (We recall that if X is actually compact, $C_0(X)$ coincides with the set of all continuous, complex-valued functions on X and is denoted by $C(X)$.)

Let A and B be any two normed spaces. For any λ, with $1 < \lambda < \infty$, we shall write $A \overset{\lambda}{\sim} B$ if there exists a linear isomorphism φ of A onto B such that $\|\varphi\| \, \|\varphi^{-1}\| < \lambda$.

The well-known Banach-Stone theorem states that if $C_0(X)$ and $C_0(Y)$ are isometrically isomorphic then X and Y are homeomorphic.

Cambern **[2]** obtained a generalization of this theorem; more precisely, he showed that if $C_0(X) \overset{2}{\sim} C_0(Y)$ then X and Y are homeomorphic. (Amir **[1]** proved this result, independently, in the special case that X and Y are compact and the functions are real-valued.)

In **[4]**, the author was able to show that in Cambern's theorem $C_0(X)$ and $C_0(Y)$ can be replaced by certain subspaces. (For another generalization of this kind see **[8]**.) To be more precise, if A and B are extremely regular subspaces of $C_0(X)$ and $C_0(Y)$ respectively, and if $A \overset{2}{\sim} B$ then X and Y are homeomorphic. (A closed linear subspace A of $C_0(X)$ is extremely regular if for each $x \in X$, each neighborhood V of x and each $0 < \varepsilon < 1$ there is an $f \in A$ such that $1 = \|f\| = f(x) > \varepsilon > |f(y)|$ for all y in $X \setminus V$. For more information about these spaces, see **[5]**.)

It is evidently seen that the main object of these results is to establish the existence of a homeomorphism of X onto Y, for locally compact spaces X and Y, provided that there is a linear isomorphism φ between certain closed linear subspaces of $C_0(X)$ and $C_0(Y)$ which is subjected to the norm condition $\|\varphi\| \, \|\varphi^{-1}\| < \lambda$. In **[3]**, Cambern showed that $\lambda = 2$ is the best number in the above generalizations. Therefore, if $\lambda > 2$ we cannot expect X and Y to be homeomorphic (except in the case that X and Y are countable compact

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[1]Throughout this paper all topological spaces are assumed to be Hausdorff.

metric spaces and $\lambda < 3$ (see [6]). However, we can at least ask whether some set-theoretic and topological properties are preserved (i.e. a property is possessed by X if, and only if, it is possessed by Y). While the axiom of second countability is clearly preserved (being equivalent to the separability of $C_0(X)$) an uncountable version of Cambern's example shows that first countability and metrizability are not.

The object of this paper is to prove the following theorem.

THEOREM. *Let X and Y be locally compact spaces. If $C_0(X) \overset{\lambda}{\sim} C_0(Y)$ for some λ then X and Y have the same cardinal number.*

Before beginning the proof of our theorem, we establish some conventions regarding notation. In what follows φ will denote a fixed linear topological isomorphism of $C_0(X)$ onto $C_0(Y)$ and φ^* its adjoint. We may assume that φ is norm-increasing (i.e. $\|f\| \leq \|\varphi f\|$ for each $f \in C_0(X)$) and that $\|\varphi^{-1}\| = 1$. (If this is not the case, we take $\psi = \|\varphi^{-1}\|\varphi$ which possesses these properties.) For any point $x \in X$, μ_x will denote the unit point mass at x. For a finite regular Borel measure μ on X, $\mu(f)$ will denote the μ-integral of f on X, $\|\mu\|$ will denote $|\mu|(X)$, where $|\mu|$ is the total variation of μ. Finally, for each y in Y, φ_y will denote the total variation of the measure $\varphi^*\mu_y$ on X, and we shall write $\varphi_y(x)$ in place of $\varphi_y(\{x\})$.

The key step in the proof of our theorem is to prove the following proposition.

(P) *For each point x in X there exists a point y in Y such that $\varphi_y(x) \neq 0$.*

Let us assume for a moment that the proposition (P) has been proved. For each $y \in Y$ let $X_y = \{x \in X\colon \varphi_y(x) \neq 0\}$. Then each X_y is countable and $X = \bigcup_{y \in Y} X_y$. Thus, we have $|X| \leq |Y|$. By exchanging the roles of X and Y we also obtain $|Y| \leq |X|$. Hence they are equal, proving our theorem.

We shall assume that (P) is false, and try to get a contradiction. Let $x \in X$ such that

$$\varphi_y(x) = 0 \quad \text{for all } y \in Y. \tag{1}$$

Let $\{U_i\colon i \in I\}$ denote the family of all open neighborhoods of x, where the index set I is assumed to be partially ordered by the relation that $j \leq i$ if, and only if, $U_i \subset U_j$. Let us fix a positive number ε and for each $i \in I$ define

$$C_i = \mathrm{Cl}\{y \in Y\colon \varphi_y(\mathrm{Cl}\, U_i) \leq \varepsilon\}.$$

Then, it follows immediately from (1) and regularity of each φ_y that

$$Y = \bigcup_{i \in I} C_i. \tag{2}$$

We claim that for any compact subset E of Y and any finite positive regular Borel measure μ on Y, $\mu(E) = 0$ provided that $\mu(E \cap C_i) = 0$ for every $i \in I$.

Let $M = \sup\|\varphi_y\|$, where the supremum is taken over all $y \in E$. Observe that for any open subset U of X, $\varphi_y(U)$ is a lower semicontinuous function of y, as the supremum of the family of continuous functions $\{|\varphi^*\mu_y(f)|: f \in C_0(U), \|f\| \leq 1\}$. Thus, for each $i \in I$, the set

$$P_i = \{y \in E: \varphi_y(V_i) > M - \varepsilon\}$$

is an open subset of the subspace E, where $V_i = X \setminus \mathrm{Cl}\ U_i$.

Let $y \in P_i$. Then

$$\|\varphi_y\| = \varphi_y(V_i) + \varphi_y(\mathrm{Cl}\ U_i) > M - \varepsilon + \varphi_y(\mathrm{Cl}\ U_i)$$

from which we get $\varphi_y(\mathrm{Cl}\ U_i) < \|\varphi_y\| - M + \varepsilon \leq \varepsilon$, and hence $P_i \subset E \cap C_i$. Consequently, $\mu(P_i) = 0$.

For each $y \in Y$, $\sup_i \varphi_y(V_i) = \|\varphi_y\|$ (by (1) and regularity of φ_y). So, if $y \in E$ is such that $\|\varphi_y\| > M - \varepsilon$ then for some $i \in I$, $\varphi_y(V_i) > M - \varepsilon$. Thus the set $A = E \setminus P_i$, where i is "sufficiently large", is a proper closed subset of E. Hence, we have constructed a proper closed subset A of E whose μ-measure is equal to that of E.

Now let us assume that $\mu(E) > 0$, and let $\mathfrak{F}$ denote the family of all proper closed subsets F of E with $\mu(F) = \mu(E)$. The relation $F_1 \leq F_2$ if, and only if, $F_2 \subset F_1$, partially orders $\mathfrak{F}$. It is not hard to see that every chain in $\mathfrak{F}$ has an upper bound. Therefore $\mathfrak{F}$ has a maximal element F_0. But then, by our discussion above, there exists a proper closed subset A of F_0 with $\mu(A) = \mu(F_0)$. Thus, $A \in \mathfrak{F}$ and $F_0 < A$ which contradicts the maximality of F_0. Hence $\mu(E) = 0$, proving our claim.

Next, we show that for any finite positive regular Borel measure μ on Y, $\mu(Y) = \sup_i \mu(C_i)$.

Let $a = \sup_i \mu(C_i)$. Since the family $\{C_i: i \in I\}$ is directed upward (i.e. $C_i \subset C_j$ whenever $i \leq j$) there exists a sequence of indexes $i_1, i_2, \ldots$ such that $C_{i_1} \subset C_{i_2} \subset \ldots$ and $a = \lim_n \mu(C_{i_n})$. Let $K = \bigcup_{n=1}^{\infty} C_{i_n}$. We shall show that $\mu(Y \setminus K) = 0$, or equivalently, $\mu(E) = 0$ for every compact subset E of $Y \setminus K$. Fix a compact set E contained in $Y \setminus K$. For each $i \in I$ and positive integer n there exists $j \in I$ such that $i \leq j$ and $i_n \leq j$. So,

$$a \geq \mu(C_j) \geq \mu(C_j \cap E) + \mu(C_{i_n})$$

from which we conclude that $\mu(C_i \cap E) = 0$, for every $i \in I$. Hence $\mu(E) = 0$.

Now we prove proposition (P). To this end, for each $i \in I$, we choose a function f_i in $C_0(X)$ such that

$$1 = \|f_i\| = f_i(x) > 0 = f(z) \quad \text{for all } z \in X \setminus U_i.$$

Then,

$$|\varphi(f_i)(y)| \leq \varphi_y(|f_i|) \leq \varphi_y(\mathrm{Cl}\ U_i)$$

for every $i \in I$ and $y \in Y$. Therefore, $|\varphi(f_i)(y)| \leq \varepsilon$ for all $y \in C_i$. Let μ denote the total variation of $\varphi^{*-1}\mu_x$ on Y. Then

$$\begin{aligned} 1 = f_i(x) &= \int \varphi(f_i)\, d\varphi^{*-1}\mu_x \\ &\leqslant \int_{C_i} |\varphi(f_i)|\, d\mu + \int_{Y \setminus C_i} |\varphi(f_i)|\, d\mu \\ &\leqslant \varepsilon\|\mu\| + \|\varphi\|\, \mu(Y \setminus C_i) \\ &\leqslant \varepsilon\|\mu\| + \|\mu\|(\|\mu\| - \mu(C_i)). \end{aligned}$$

The infimum of this last expression over i is $\varepsilon\|\mu\|$ (by the preceding result), and so we obtain $1 \leqslant \varepsilon\|\mu\|$, which is absurd since ε was arbitrary. This contradiction completes the proof of (P).

REMARK. The condition $|X| = |Y|$ is far from being sufficient (except in the case that X and Y are both uncountable compact metric spaces (see [7])) for $C_0(X)$ and $C_0(Y)$ to be isomorphic. (Let D be a discrete space with $|D| = |\beta N| = 2^c$, where N is the discrete space of positive integers, c is the cardinal number of the set of real numbers, and where βN denotes the Stone-Čech compactification of N. Then $|C_0(D)| = |D| = 2^c > c = |C(\beta N)|$.)

REFERENCES

1. D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. **3** (1965), 205–210.
2. M. Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc. **18** (1967), 1062–1066.
3. ______, *Isomorphisms of $C_0(Y)$ onto $C(X)$*, Pacific J. Math. **35** (1970), 307–312.
4. B. Cengiz, *A generalization of the Banach-Stone theorem*, Proc. Amer. Math. Soc. **40** (1973), 426–430.
5. ______, *On extremely regular function spaces*, Pacific J. Math. **49** (1973), 335–338.
6. Y. Gordon, *On the distance coefficient between isomorphic function spaces*, Israel J. Math. **8** (1970), 391–397.
7. A. A. Milutin, *Isomorphisms of the spaces of continuous functions over compact sets of cardinality of the continuum*, Teor. Funkcii Funkcional Anal. i Priložen. **2** (1966), 150–156.
8. S. B. Myers, *Banach spaces of continuous functions*, Ann. of Math. (2) **49** (1948), 132–140.

DEPARTMENT OF MATHEMATICS, MIDDLE EAST TECHNICAL UNIVERSITY, ANKARA, TURKEY