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A CLASS OF MAPPINGS CONTAINING ALL CONTINUOUS AND ALL SEMICONNECTED MAPPINGS

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Abstract. A function f: $X \to Y$ is called s-continuous if for each $x \in X$ and each open set V containing $f(x)$ and having connected complement there is an open set U containing x such that $f(U) \subset V$. In this paper basic properties of s-continuous functions are studied; conditions on domain and/or range implying continuity of s-continuous functions are obtained which generalize recent theorems of Jones, Lee and Long on semiconnected functions. Improvements of recent results of Hagan, Kohli and Long concerning the continuity of certain connected functions follow as a consequence. Also characterizations of semilocally connected spaces in terms of s-continuous functions are obtained.

1. Introduction. A function f: $X \to Y$ is called

(i) *semiconnected* if for each closed and connected set $K \subset Y, f^{-1}(K)$ is closed and connected;

(ii) *weak semiconnected* if for each closed and connected set $K \subset Y, f^{-1}(K)$ is closed;

(iii) *connected* if $f(A)$ is connected for each connected set $A \subset X$; and

(iv) *monotone* if for each $y \in Y, f^{-1}(y)$ is connected.

Several authors have studied semiconnected functions (see Lee [**8**], Jones [**5**], and Long [**9**]) and weak semiconnected functions are considered in ([**6**], [**7**]). The concepts of continuity and semiconnectedness are independent of each other and both imply weak semiconnectedness. However, a weak semiconnected function need not be either continuous or semiconnected. For example, let R_1 and R_2 denote the real line endowed with usual and discrete topologies, respectively, and let X denote the disjoint topological sum of R_1 and R_2. Let f: $X \to X$ be the function whose restriction to R_1 is the identity mapping from R_1 onto R_2 and whose restriction to R_2 is the identity mapping from R_2 onto R_1. Then f is a weak semiconnected function which is neither continuous nor semiconnected.

2. Basic properties of s-continuous functions.

2.1 Theorem. *Let f: $X \to Y$ be a function from a topological space X into a topological space Y. The following statements are equivalent*:

(a) *f is s-continuous*.

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(b) *If V is an open subset of Y with connected complement, then $f^{-1}(V)$ is an open subset of X.*

(c) *f is weak semiconnected.*

PROOF. (a) $\Rightarrow$ (b). If V is an open subset of Y with connected complement, then for each $x \in f^{-1}(V)$, V is a neighbourhood of $f(x)$. Hence there is a neighbourhood U of x such that $f(U) \subset V$. Thus $f^{-1}(V)$ being a neighbourhood of each of its points is open.

(b) $\Rightarrow$ (a). Let $x \in X$ and let V be an open set containing $f(x)$ and having connected complement. Then $f^{-1}(V)$ is an open set containing x and $f(f^{-1}(V)) \subset V$.

(b) $\Rightarrow$ (c). Let $K \subset Y$ be a closed connected set. Then $Y - K$ is an open set with connected complement and, therefore, $f^{-1}(Y - K) = X - f^{-1}(K)$ is open. So, $f^{-1}(K)$ is closed in X.

(c) $\Rightarrow$ (b). Let $V \subset Y$ be an open set with connected complement.

Then $Y - V$ is a closed connected set and, therefore $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed. So, $f^{-1}(V)$ is open in X.

2.2 COROLLARY. *Let f: $X \to Y$ be s-continuous and injective. If Y is T_1, so is X.*

2.3 THEOREM. *Let f: $X \to Y$ be a s-continuous, closed function from a normal space X onto a space Y. If either of the spaces X and Y is T_1, then Y is Hausdorff.*

PROOF. *Case* I. The space Y is T_1. Let y_1, y_2 be any two distinct points in Y. Then $\{y_1\}$ and $\{y_2\}$ are closed connected subsets of Y so that by Theorem 2.1, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed subsets of X. By normality of X, there are disjoint open sets U_1 and U_2 containing $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since f is closed, the sets $V_1 = Y - f(X - U_1)$ and $V_2 = Y - f(X - U_2)$ are open in Y. It is easily verified that V_1 and V_2 are disjoint and contain y_1 and y_2, respectively. Thus Y is Hausdorff.

Case II. The space X is T_1. Let $f(x) \in Y$ be any point. Since the singleton $\{x\}$ is closed in X, $\{f(x)\}$ is a closed subset of Y. So Y is T_1 and the proof is complete in view of Case I.

2.4 COROLLARY. *Every continuous closed image of a normal T_1-space is T_2 and hence T_4.*

PROOF. Every continuous closed image of a normal space is normal [**2**, p. 154].

2.5 COROLLARY. *Every continuous closed image of a compact Hausdorff space is compact Hausdorff.*

2.1 REMARK. Theorem 2.3 is false with 'closed' replaced by 'open'. For let X be the union of the lines $y = 0$ and $y = 1$ in the euclidean plane and let Y be the quotient of X obtained by identifying each point $(x, 0)$ for $x \neq 0$, with the corresponding point $(x, 1)$. The resulting quotient map q: $X \to Y$ is continuous open, and Y is T_1, but $q(0, 0)$ and $q(0, 1)$ are distinct points of Y which do not have disjoint neighbourhoods.

2.6 THEOREM. *Let f: $X \to Y$ be s-continuous and let Y be a locally connected T_3-space. Then f has closed graph.*[1]

PROOF. Let (x, y) be any point in $X \times Y$ which does not lie in the graph of f. Then $f(x) \neq y$. Since Y is T_3 and hence T_2, there are disjoint open sets V_1 and V_2 containing $f(x)$ and y, respectively. Since Y is a locally connected T_3-space, there exists an open connected set V such that $y \in V \subset \overline{V} \subset V_2$. By Theorem 2.1, $f^{-1}(\overline{V})$ is closed in X and does not contain x. Since f is s-continuous, there is an open set U such that $x \in U \subset X - f^{-1}(\overline{V})$ and such that $f(U) \subset Y - \overline{V}$. Therefore $U \times V$ contains (x, y) but no point of $G(f)$. Thus $G(f)$ is closed in $X \times Y$.

2.1 DEFINITION. Let f: $X \to Y$ be any function. Then the function g: $X \to X \times Y$, defined by $g(x) = (x, f(x))$, is called the *graph function* with respect to f. There are certain relationships between a function and its graph function.

2.7 THEOREM.[2] *If f: $X \to Y$ is a function from a connected space X into a space Y such that the graph function is s-continuous, then f is s-continuous.*

PROOF. Let $x \in X$ and V be an open set containing $f(x)$ such that $Y - V$ is connected. Then $p_Y^{-1}(V)$ is open in $X \times Y$. Since X and $Y - V$ are connected, $X \times (Y - V) = (X \times Y) - p_Y^{-1}(Y)$ is connected. Thus $p_Y^{-1}(V)$ is an open set in $X \times Y$ having a connected complement. Therefore, there exists an open set U containing x such that $g(U) \subset p_Y^{-1}(V)$. It follows that $p_Y(g(U)) = f(U) \subset V$, so that f is s-continuous.

2.8 THEOREM. *Let f: $X \to Y$ be any function. Then the following statements are true.*

(a) *If f is s-continuous and $A \subset X$, then $f|A$: $A \to Y$ is s-continuous.*

(b) *If $\{U_\alpha$: $\alpha \in A\}$ is an open cover of X and if for each α, $f_\alpha = f|U_\alpha$ is s-continuous, then f is s-continuous.*

(c) *If $\{F_\beta$: $\beta \in B\}$ is a locally finite closed cover of X and if for each β, $f_\beta = f|F_\beta$ is s-continuous, then f is s-continuous.*

PROOF. (a) Let U be an open subset of Y with connected complement. Then $f^{-1}(U)$ is open and hence $(f|A)^{-1}(U) = f^{-1}(U) \cap A$ is an open subset of A.

(b) Let U be an open subset of Y with connected complement. Then

[1]The referee pointed out that graph is actually strongly closed in the sense of **[10]**.

[2]The interesting problem that whether the converse of Theorem 2.7 is true remains open and was raised by the referee.

$f^{-1}(U) = \bigcup\{f_\alpha^{-1}(U)\colon \alpha \in A\}$ and since each f_α is s-continuous, each $f_\alpha^{-1}(U)$ is open in X and so $f^{-1}(U)$ is open in X.

(c) Let F be a closed and connected subset of Y. Then $f^{-1}(F) = \bigcup\{f_\beta^{-1}(F)\colon \beta \in B\}$. Since each f_β is s-continuous, each $f_\beta^{-1}(F)$ is closed in F_β and hence in X. Again, since $\{F_\beta\colon \beta \in B\}$ is a locally finite closed cover of X, the collection $\{f_\beta^{-1}(F)\colon \beta \in B\}$ is a locally finite collection of closed sets. Thus $f^{-1}(F)$ being the union of a locally finite collection of closed sets is closed [**1**, p 22].

2.9 THEOREM. *If $f\colon X \to Y$ is continuous and $g\colon Y \to Z$ is s-continuous, then $g \circ f\colon X \to Z$ is s-continuous.*

PROOF. Let K be a closed and connected subset of Z. Then $g^{-1}(K)$ is closed and since f is continuous, $(g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K))$ is closed.

2.2 REMARK. If f is s-continuous and g is continuous, then in general $g \circ f$ need not be s-continuous. For, let X, Y and Z denote the real line endowed with cofinite topology, discrete topology and usual topology, respectively. Let $f\colon X \to Y$ and $g\colon Y \to Z$ denote the identity maps. Then $g \circ f$ is not s-continuous. Thus, in particular, composition of two s-continuous functions may fail to be s-continuous.

2.10 THEOREM. *Let $f\colon X \to Y$ be a quotient map. Then a function $g\colon Y \to Z$ is s-continuous if and only if $g \circ f$ is s-continuous.*

PROOF. Necessity follows from Theorem 2.9. To prove sufficiency, let U be an open subset of Z with connected complement. Then

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$$

is open in X. Since f is a quotient map, $g^{-1}(U)$ is open in Y so g is s-continuous.

2.2 DEFINITION [**3**]. A topological space X is called a *saturated space* if any intersection of open sets in X is itself an open set; or equivalently every point of X possesses a minimum neighbourhood.

2.11 THEOREM. *Let X be a saturated space and let Y be a locally connected regular space. If $f\colon X \to Y$ is s-continuous, then f is continuous.*

PROOF. Let $x \in X$ and let V be an open subset of Y containing $f(x)$. Since Y is regular, there is an open set U such that $f(x) \in U \subset \overline{U} \subset V$. Let $y \in Y - \overline{U}$. Since Y is regular, there is an open set V_y containing y such that $V_y \cap \overline{U} = \varnothing$. Since Y is also locally connected, there exists a closed connected neighbourhood U_y of y such that $U_y \cap U = \varnothing$. Thus $Y - U_y$ is an open set containing $f(x)$ and has connected complement. Since f is s-continuous, there is an open set N_y containing x such that $f(N_y) \subset Y - U_y$. Let $N = \bigcap\{N_y\colon y \in Y - \overline{U}\}$. Now N contains x and since X is a saturated space, N is open. Clearly, $f(N) \subset \overline{U} \subset V$ and hence f is continuous.

2.3 DEFINITION. A topological space X is called *semilocally connected* if for each $x \in X$ and each open set U containing x there is an open set V such that $x \in V \subset U$ and $X - V$ consists of a finite number of components.

2.3 REMARK. The above definition of semilocal connectedness differs from the one that occurs in the literature (see **[8]**–**[11]**) in the sense that we do not necessarily require a semilocally connected space to be a connected T_1-space. Thus every finite topological space as well as every indiscrete space is semilocally connected.

2.12 THEOREM. *If f: $X \to Y$ is s-continuous and if Y is semilocally connected, then f is continuous.*

PROOF. Let $x \in X$ and let V be any open neighbourhood of $f(x) = y$ in Y. Since Y is semilocally connected, there is an open set $N_y \subset V$ containing y such that $Y - N_y$ consists of a finite number of components $C_1, C_2, \ldots, C_n$. For each $k = 1, \ldots, n$, C_k is closed and connected so that $f^{-1}(C_k)$ is closed by Theorem 2.1. Therefore, $\bigcup_{i=1}^{n} f^{-1}(C_k) = A$ is a closed subset of X and does not contain a point of $f^{-1}(y)$. So, $U = X - A$ is an open set containing x and $f(U) = N_y \subset V$. Thus f is continuous.

2.13 COROLLARY (LEE **[8]**, LONG **[9]**). *If f: $X \to Y$ is semiconnected and if Y is semilocally connected, then f is continuous.*

2.14 COROLLARY (KOHLI **[6]**). *Let f: $X \to Y$ be a closed (or open) connected monotone function and Y and T_1-space. If Y is semilocally connected then f is continuous.*

PROOF. It is easily verified that an open (or a closed) connected monotone function into a T_1-space is semiconnected. Therefore, by Corollary 2.13, f is continuous.

2.4 REMARK. Corollary 2.14 generalizes Theorems 1 and 7 of Hagan **[4]** and also includes Corollary 2 of **[9]**.

3. Characterizations of semilocally connected spaces. Let (X, τ) be a topological space and let $\mathbb{S}$ denote the collection of all open sets whose complements are connected. Let τ^* denote the topology on X generated by taking $\mathbb{S}$ as a subbase. Obviously, $\tau^* \subset \tau$. Further it is easily verified that if (X, τ) is T_1, or compact, or connected, so is (X, τ^*). In the sequel that follows, τ^* will always have the same meaning as in this paragraph.

3.1 THEOREM. *The space (X, τ^*) is semilocally connected.*

PROOF. Since $\tau^* \subset \tau$, every τ-connected set is τ^*-connected. Let $\mathfrak{B}$ be the collection of all finite intersections of members of $\mathbb{S}$ and let $B \in \mathfrak{B}$. Then $B = \bigcap_{i=1}^{n} U_i$, where each $U_i \in \mathbb{S}$ and hence $Y - B = \bigcup_{i=1}^{n}(Y - U_i)$. Since each $Y - U_i$ is τ-connected, it is τ^*-connected. Thus τ^* has a base such that the complement of each basic open set consists of a finite number of components and so (X, τ^*) is semilocally connected.

3.2 COROLLARY. *Any topological space X can be condensed onto a semilocally* *connected space Y. Moreover, if X is compact, or connected, or T_1, so is Y.*

3.3 THEOREM. *The space (X, τ) is semilocally connected if and only if $\tau = \tau^*$.*

PROOF. Sufficiency follows from Theorem 3.1. To prove necessity, suppose that (X, τ) is semilocally connected and let $x \in U$. Let U be a τ-open set containing x and such that $X - U$ consists of a finite number of components $C_1, \ldots, C_n$. For each $k = 1, 2, \ldots, n$, C_k is closed and connected so that $Y - C_k \in \mathcal{S}$. Thus

$$\bigcap_{k=1}^{n} (Y - C_k) = Y - \bigcup_{k=1}^{n} C_k = U$$

is a τ^*-basic open set and hence $\tau \subset \tau^*$. So, $\tau = \tau^*$.

3.4 COROLLARY. *The operator $\tau \to \tau^*$ is idempotent, i.e., $(\tau^*)^* = \tau$ for every topological space (X, τ).*

3.1 REMARK. In general τ^* need not be either the finest or the coarsest semilocally connected topology contained in τ. For, let $\mathcal{U}$, $\mathcal{D}$ and $\mathcal{C}$ denote the usual, discrete and cofinite topologies, respectively on the real line $\mathbf{R}$. Now, if $(X, \tau) = (\mathbf{R}, \mathcal{D})$, then $\tau^* = \mathcal{C} \subsetneq \mathcal{U}$; and if τ denotes the topology on $\mathbf{R}$ generated by taking $\{Q\} \cup \mathcal{U}$ as a subbase, where Q is the set of rationals, then $\tau^* = \mathcal{U} \supsetneq \mathcal{C}$.

3.5 THEOREM. *Let (X, τ) be a topological space. Then the following statements are equivalent*:

(a) *(X, τ) is semilocally connected.*

(b) *Every s-continuous function f from a topological space Y into (X, τ) is continuous.*

(c) *Every semiconnected function f from a topological space Y into (X, τ) is continuous.*

(d) *The identity mapping 1_X from (X, τ^*) onto (X, τ) is continuous.*

PROOF. (a) $\Rightarrow$ (b) follows from Theorem 2.12 and (b) $\Rightarrow$ (c) is obvious. The implication (c) $\Rightarrow$ (d) follows, since the identity mapping 1_X is semiconnected. The implication (d) $\Rightarrow$ (a) is immediate in view of Theorem 3.1 and the fact that $\tau^* \subset \tau$.

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