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APPROXIMATION OF L^1-BOUNDED MARTINGALES BY MARTINGALES OF BOUNDED VARIATION

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ABSTRACT. If $f = (f_1, f_2, \ldots)$ is a real L^1-bounded martingale and $\varepsilon > 0$, then there is a martingale g of bounded variation satisfying $\|f - g\|_1 < \varepsilon$. The same result holds for X-valued martingales, where X is a Banach space, provided X has the Radon-Nikodým property. In fact, this characterizes Banach spaces having the Radon-Nikodým property. Theorem 1 identifies, for an arbitrary Banach space, the class of L^1-bounded martingales that converge almost everywhere.

Let X be a real or complex Banach space with norm $|\cdot|$, $(\Omega, \mathcal{Q}, P)$ a probability space, and $L^1 = L^1(X)$ the space of strongly integrable functions $f\colon \Omega \to X$ with norm $\|f\|_1 = E|f|$. Here, and throughout the paper, $|f| = |f(\cdot)|$ and E denotes expectation: integration over Ω with respect to P. Consider $M^1 = M^1(X)$, the space of $L^1(X)$-bounded martingales $f = (f_1, f_2, \ldots)$ relative to a fixed increasing sequence $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ of sub-σ-fields of $\mathcal{Q}$. Equipped with the norm $\|f\|_1 = \sup_n \|f_n\|_1$, M^1 is a Banach space.

A martingale f, with $f_n = \sum_{k=1}^n d_k$, $n \geqslant 1$, is of *bounded variation* if $\sum_{k=1}^\infty |d_k| < \infty$ a.e. Note that this is quite different, and weaker, than requiring that $\sum_{k=1}^\infty \|d_k\|_1 < \infty$. Let

$$BV = \{f \in M^1 : f \text{ is of bounded variation}\},$$

$$AE = \{f \in M^1 : f \text{ converges a.e.}\},$$

and BV^- denote the closure in M^1 of the class BV. Clearly, $BV \subset AE$; what is not quite so transparent is that BV is dense in AE:

THEOREM 1. $BV^- = AE$.

PROOF. $BV^- \subset AE$: Suppose $f \in BV^-$ and $\varepsilon > 0$. Then there is a $g \in BV$ such that $\|f - g\|_1 < \varepsilon^2$. Therefore, by the vector-valued version of a classical martingale inequality [**2**, p. 128],

$$P\Big(\sup_n |f_n - g_n| > \varepsilon\Big) \leqslant \varepsilon^{-1}\|f - g\|_1 < \varepsilon$$

and, since g converges almost everywhere,

$$P\Big(\limsup_{m,n\to\infty} |f_m - f_n| > 2\varepsilon\Big) < \varepsilon.$$

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This implies that $f \in AE$.

$AE \subset BV^-$: Let $f \in AE$ and f_∞ denote the almost everywhere pointwise limit of f. By Fatou's lemma, $f_\infty \in L^1(X)$. Write $f = g + h$ where $g_n = E(f_\infty|\mathcal{Q}_n)$, the conditional expectation of f_∞ given $\mathcal{Q}_n$ for $n \geqslant 1$. Note that both g and h belong to M^1, and, by the continuity theorem for conditional expectations, $g_n \to f_\infty$ a.e. and $\|g_n - f_\infty\|_1 \to 0$ as $n \to \infty$ (for example, see **[1**, pp. 27 and 23]).

Let $\varepsilon > 0$. There exists a stopping time τ, finite almost everywhere, such that the stopped martingale $f^\tau = (f_{\tau\wedge 1}, f_{\tau\wedge 2}, \dots)$ satisfies $\|f - f^\tau\|_1 < \varepsilon$. Since τ is finite almost everywhere, $f^\tau \in BV$ implying $AE \subset BV^-$, the desired result. To obtain τ, we work with the smallest martingale majorant (Snell, Krickeberg) of the submartingale $(|h_1|, |h_2|, \dots)$. Let $H_n = \sup_{k \geqslant n} E(|h_k|\,|\mathcal{Q}_n)$ and note that $H_n \geqslant |h_n|$. The fact that $E(|h_k|\,|\mathcal{Q}_n)$ is nondecreasing in k implies that $H = (H_1, H_2, \dots)$ is an $L^1(\mathbf{R})$-bounded martingale satisfying $\|H\|_1 = \|h\|_1$. Since $h_n = f_n - g_n \to f_\infty - f_\infty = 0$ a.e.,

$$E \liminf H_n \leqslant \liminf E(H_n - |h_n|) = \|H\|_1 - \|h\|_1 = 0.$$

Therefore, $\liminf H_n = 0$ a.e. and the stopping time τ_j, defined by

$$\tau_j(\omega) = \inf\{n\colon \text{exactly } j \text{ of } H_1(\omega), \dots, H_n(\omega) \text{ are less than } 2^{-j}\},$$

is finite almost everywhere.

Now let $\tau = \tau_j$ where j is a positive integer satisfying $2^{-j} < \varepsilon/4$ and $\|g_n - f_\infty\|_1 < \varepsilon/8$ for all $n \geqslant j$. Then, as we show below, $\|g - g^\tau\|_1 \leqslant \varepsilon/2$ and $\|h - h^\tau\|_1 < \varepsilon/2$ implying that $\|f - f^\tau\|_1 < \varepsilon$. By its definition, the stopping time τ satisfies $\tau \geqslant j$. So, if $n \geqslant j$, then $\tau \wedge n \geqslant j$ and $\|g_{\tau\wedge n} - g_j\|_1 \leqslant \|g_n - g_j\|_1 < \varepsilon/4$. Therefore,

$$\|g - g^\tau\|_1 = \lim_{n\to\infty} \|g_n - g_{\tau\wedge n}\|_1 \leqslant \varepsilon/2.$$

The corresponding inequality for h follows from

$$|h_n - h_{\tau\wedge n}| \leqslant H_n - H_{\tau\wedge n} + 2a \tag{1}$$

where $a = 2^{-j}$. To prove (1), note that if $n \leqslant \tau$, then (1) reduces to $0 \leqslant 2a$; if $\tau < n$, then

$$\begin{aligned}|h_n - h_\tau| &\leqslant |h_n| + |h_\tau| \leqslant H_n + H_\tau \\ &= H_n - H_\tau + 2H_\tau \leqslant H_n - H_\tau + 2a,\end{aligned}$$

and (1) holds in this case also. Taking expectations of both sides of (1) and using the fact that $EH_{\tau\wedge n} = EH_1 = EH_n$ (each term of a martingale has the same expectation), we obtain $\|h_n - h_{\tau\wedge n}\|_1 \leqslant 2a$. Therefore, $\|h - h^\tau\|_1 \leqslant 2a < \varepsilon/2$. This completes the proof of Theorem 1.

If $X = \mathbf{R}$, then $AE = M^1$ by the classical martingale convergence theorem of Doob. So we have the following equivalent statement of the result in the real case.

COROLLARY. *If $f = (f_1, f_2, \dots)$ is a real L^1-bounded martingale and* $\varepsilon > 0$, *then there is a martingale g of bounded variation, g adapted to the same sequence of sub-σ-fields as f, such that $\|f - g\|_1 < \varepsilon$.*

For what kind of a Banach space is such an approximation always possible? To answer this question, we combine Theorem 1 with a result of Chatterji:

THEOREM 2. *For a Banach space X, the following are equivalent*:
(i) $BV^- = M^1$ *for all* $(\Omega, \mathcal{Q}, P)$ *and* $\mathcal{Q}_1, \mathcal{Q}_2, \ldots,$
(ii) $AE = M^1$ *for all* $(\Omega, \mathcal{Q}, P)$ *and* $\mathcal{Q}_1, \mathcal{Q}_2, \ldots,$
(iii) *X has the Radon-Nikodým property.*

PROOF. Theorem 1 implies the equivalence of (i) and (ii); Chatterji's result [**1**, p. 31] is the equivalence of (ii) and (iii).

For a number of other properties equivalent to the Radon-Nikodým property, see the book by Diestel and Uhl [**2**, pp. 217–219].

A final question: Is approximation possible for martingales indexed by continuous time? Even in the real case, the answer is generally negative. Let $B = \{B_t, 0 \leqslant t \leqslant 1\}$ be a real Brownian motion and $\mathcal{B}_t$ the smallest σ-field with respect to which B_s is measurable for all $s \leqslant t$.

THEOREM 3. *Suppose that $Y = \{Y_t, 0 \leqslant t \leqslant 1\}$ is a martingale adapted to $\{\mathcal{B}_t, 0 \leqslant t \leqslant 1\}$ such that $\|B - Y\|_1 < \|B\|_1$. Then Y is not of bounded variation.*

This is intuitively clear. Here is one of the several possible proofs.

PROOF. Suppose that Y is of bounded variation: for almost all ω, the mapping $t \to Y_t(\omega)$ is a function of bounded variation on [0, 1]. By a result of Dudley [**3**], there exists an Itô integral $X_1 = \int_0^1 \varphi(t)\, dB_t$ such that $P(X_1 = Y_1) = 1$. Consider the martingale $X = \{X_t, 0 \leqslant t \leqslant 1\}$ with continuous paths defined by $X_t = \int_0^t \varphi(s)\, dB_s$. The martingale $X - Y$ satisfies $E|X_t - Y_t| \leqslant E|X_1 - Y_1| = 0$ so $P(X_t = Y_t) = 1$, $0 \leqslant t \leqslant 1$. This implies that X is of bounded variation on the rationals so, by continuity, X is of bounded variation. As an Itô integral, X may be written as a new Brownian motion with a change of time [**4**, p. 29] and, since Brownian motion is not of bounded variation, $P(\sup_{0 \leqslant t \leqslant 1}|X_t| = 0) = 1$. Accordingly, $\|B - Y\|_1 = E|B_1 - Y_1| = E|B_1 - X_1| = E|B_1| = \|B\|_1$. But this is a contradiction and the theorem is proved.

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