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L^2-BOUNDEDNESS FOR PSEUDO-DIFFERENTIAL OPERATORS WITH UNBOUNDED SYMBOLS

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ABSTRACT. Kato has proven L^2-boundedness if the symbol $a(x, z)$ is such that $|D_x^\beta D_z^\alpha a(x, z)| < (\text{constant})(1 + |z|)^{(|\beta|-|\alpha|)\rho}$ for $|\alpha| < [n/2] + 1$, $|\beta| < [n/2] + 2$ and $0 < \rho < 1$. In this paper, L^2-boundedness is shown for a corresponding Hölder continuity condition which requires slightly less smoothness for $a(x, z)$.

In [1], Calderón and Vaillancourt prove L^2-boundedness for symbols $a(x_1, x_2, z)$ of three variables under conditions which for the special case $a(x, z)$ reduce to

$$|D_x^\beta D_z^\alpha a(x, z)| \leqslant (\text{constant})(1 + |z|)^{(|\beta|-|\alpha|)\rho}$$

whenever $|\alpha| \leqslant 2[n/2] + 2$ and $|\beta| \leqslant [5n/4(1 - \rho)] + 1$ where $0 < \rho < 1$. Kato, in [2], establishes L^2-boundedness even if the condition is only satisfied for $|\alpha| \leqslant [n/2] + 1$ and $|\beta| \leqslant [n/2] + 2$. He was motivated by and used the method of Cordes in [3]. Cordes proved L^2-boundedness if the condition is satisfied for $|\alpha| \leqslant [n/2] + 1$ and $|\beta| \leqslant [n/2] + 1$, but with $\rho = 0$. That is, Cordes' symbols were bounded. We expand Kato's result, needing only a Hölder continuity condition which will require slightly less smoothness for $a(x, z)$. This, unfortunately, may be somewhat obscured by the fact that we look at $a(x, z)$ as being a function of $2n$ real variables rather than 2 n-vector variables. This different point of view is used in the original paper of Calderón and Vaillancourt [4] and also in [5], which are results for bounded symbols. Our paper makes reasonably available results previously given in [6].

To state our result, we must introduce certain notations.

DEFINITION. Let a be a complex-valued function on R^n. Then, the shift operator is defined by

$$(S_h^\alpha a)(x_1, \ldots, x_n) = a(x_1 + \alpha_1 h_1, \ldots, x_n + \alpha_n h_n)$$

where $h \in R^n$ and α is a multi-index, all of whose entries, α_i, are either 0 or 1. For the shift operator (at most) one of the entries will be 1 with the rest 0. The difference operator is defined by

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$$D_h^\alpha = \begin{cases} \text{identity} & \text{for } |\alpha| = 0, \\ S_h^\alpha\text{-identity} & \text{for } |\alpha| = 1, \\ D_h^{\alpha^1} \cdots D_h^{\alpha^n} & \text{for } |\alpha| \geqslant 2, \end{cases}$$

where α^i is a multi-index with all entries 0 except possibly the ith which has the value of α_i. The differential-difference operator is defined by

$$\Delta_h^\alpha = \begin{cases} \text{identity} & \text{for } |\alpha| = 0, \\ S_h^\alpha\text{-identity} - h_i \partial/\partial x_i & \text{for } |\alpha| = 1, \\ \Delta_h^{\alpha^1} \cdots \Delta_h^{\alpha^n} & \text{for } |\alpha| > 2, \end{cases}$$

where the i index in the definition for $|\alpha| = 1$ is the index such that $\alpha_i = 1$.

We can state our main result

THEOREM. *Let $a(x, z)$ be a complex-valued function defined on $R^n \times R^n$. Suppose there exist constants $c_{\alpha\beta}$, $0 < \rho < 1$, and $\delta > 0$ such that for sufficiently small $(h, k) \in R^{2n}$ we have*

$$|\Delta_h^\beta D_k^\alpha a(x, z)| \leqslant c_{\alpha\beta}(1 + |z|)^{\rho[|\beta|(3/2+\delta) - |\alpha|(1/2+\delta)]} \cdot |h_1|^{3/2+\delta} \cdots |h_n|^{3/2+\delta} |k_1|^{1/2+\delta} \cdots |k_n|^{1/2+\delta} \tag{1}$$

for all multi-indices with entries restricted to the values: 0 *or* 1. (*Note*: Δ_h^β *acts on x and* D_k^α *acts on z*.) *Then, the pseudo-differential operator defined by*

$$(Au)(x) = (2\pi)^{-n/2} \int_{R^n} e^{ix\cdot z} a(x, z) \hat{u}(z)\, dz$$

for all $u \in \mathcal{S}(R^n)$, the Schwartz space on R^n, is uniquely extendable to be a bounded linear operator on $L^2(R^n)$.

PROOF. We almost completely follow Kato's proof in [**2**]. The only exception is to establish that a certain lemma which Kato needs can be proved with our weaker smoothness condition.

The radial partition of unity of Kato is used: Let $\{\phi_j; j = 1, 2, 3, \ldots\}$ be any partition of unity: $\Sigma\phi_j = 1$ on $[0, \infty)$ with the following properties. $\phi_1 \in C_0^\infty[0, \infty)$ with $\phi_1(r) = 1$ for $0 \leqslant r \leqslant 1$. If $j \geqslant 2$, $\phi_j \in C_0^\infty(0, \infty)$ with support in $[j-1, j+1]$ and $\phi_j(j + r) = \phi_2(2 + r)$. Note that all the derivatives of the ϕ_j are uniformly bounded with respect to j. Let $|z|_*$ be a C^∞ function of $z \in R^n$ such that $|z|_* = |z|$ for $|z| > 1$ and $0 < |z|_* < 1$ for $|z| < 1$. Set $\Phi_j(z) = \phi_j(|z|_*^{1-\rho})$. Then $\{\Phi_j\}$ is a partition of unity on R^n, with $\Phi_j \in C_0^\infty(R^n)$ and $\Phi_j(z) = \phi_j(|z|^{1-\rho})$ for $j \geqslant 2$.

Kato gives the following results:

$$(j/2)^{1/(1-\rho)} \leqslant |z| \leqslant (2j)^{1/(1-\rho)}, \qquad j \geqslant 2, \tag{2}$$

$$\left||z| - j^{1/(1-\rho)}\right| \leqslant c_2 j^{\rho/(1-\rho)}, \qquad j \geqslant 1, \tag{3}$$

for $z \in \operatorname{supp} \Phi_j$. It is easy to get, in addition,

$$\left|D_k^\alpha \Phi_j(z)\right| \leqslant c_1 |k_1|^{\alpha_1(1/2+\delta)} \cdots |k_n|^{\alpha_n(1/2+\delta)} \left(|z| - \sqrt{|\alpha|}\, \varepsilon'\right)^{-\rho|\alpha|}, \qquad j \geqslant 2 \quad (4)$$

(where $\varepsilon' > 0$ is such that $\|(h, k)\| \leqslant \varepsilon' \leqslant 1/\sqrt{n}$ and is the "sufficiently small" mentioned in the theorem statement).

We set

$$a_j(x, z) = \Phi_j(z) a(x, z), \quad j = 1, 2, 3, \ldots, \qquad (5)$$

so that $a(x, z) = \Sigma_j a_j(x, z)$. Using (1), (2), (4), and (5) one obtains (after some effort)

$$\begin{aligned} \left|\Delta_h^\beta D_k^\alpha a_j(x, z)\right| \leqslant\ & c_3 \chi_j(z) |h_1|^{\beta_1(3/2+\delta)} \cdots |h_n|^{\beta_n(3/2+\delta)} \\ & \cdot |k_1|^{\alpha_1(1/2+\delta)} \cdots |k_n|^{\alpha_n(1/2+\delta)} j^{\sigma(|\beta|(3/2+\delta) - |\alpha|(1/2+\delta))}, \end{aligned} \quad (6)$$

$\sigma = \rho/(1 - \rho)$, where χ_j is the characteristic function of the support of Φ_j.

From (3), we have $\chi_j(z) \leqslant \chi'_j(j^{-\sigma} z)$where

$$\chi'_j(z) = \begin{cases} 1 & \text{for } j - c_2 \leqslant |z| \leqslant j + c_2, \\ 0 & \text{otherwise.} \end{cases}$$

Set $a_j(x, z) = a_j(j^{-\sigma} x, j^\sigma z)$. Then (6) becomes

$$\begin{aligned} \left|\Delta_h^\beta D_k^\alpha a'_j(x, z)\right| \leqslant\ & c_3 \chi'_j(z) |h_1|^{\beta_1(3/2+\delta)} \cdots |h_n|^{\beta_n(3/2+\delta)} \\ & \cdot |k_1|^{\alpha_1(1/2+\delta)} \cdots |k_n|^{\alpha_n(1/2+\delta)}. \end{aligned} \quad (7)$$

Define

$$\begin{aligned} b'_j(x, z) = & \left(1 - \frac{\partial^2}{\partial x_1^2}\right)^{3/4+\varepsilon} \cdots \left(1 - \frac{\partial^2}{\partial x_n^2}\right)^{3/4+\varepsilon} \\ & \cdot \left(1 - \frac{\partial^2}{\partial z_1^2}\right)^{1/4+\varepsilon} \cdots \left(1 - \frac{\partial^2}{\partial z_n^2}\right)^{1/4+\varepsilon} a'_j(x, z) \end{aligned}$$

where ε is some fixed number, $0 < \varepsilon < \delta/2$. We need the

LEMMA. *There is a finite, positive measure μ on R^n such that*

$$|b'_j(x, z)| \leqslant (\mu * \chi'_j)(z) \equiv \omega'_j(z), \qquad \int_{\mathbf{R}^n} |z|\, d\mu(z) < \infty.$$

OUTLINE OF PROOF. The complete proof of this Lemma is quite lengthy and not particularly enlightening. It is found in [**6**]. A very similar theorem is proved in [**5**].

What is central is to show that b'_j has the more concrete representation

$$b_j'(x,z) = a_j'(x,z) + \sum_{|\alpha|=1}^{n} \int_{R^{|\alpha|}} D_{w-z}^{\alpha} a(x,z) \prod_{j=1}^{|\alpha|} \psi_{-1/4-\varepsilon}(z_{i_j} - w_{i_j})\, dw_{i_j}$$

$$+ \sum_{|\beta|=1}^{n} \int_{R^{|\beta|}} \Delta_{y-x}^{\beta} a(x,z) \prod_{k=1}^{|\beta|} \psi_{-3/4-\varepsilon}(x_{i_k} - y_{i_k})\, dy_{i_k}$$

$$+ \sum_{|\alpha|=1}^{n} \sum_{|\beta|=1}^{n} \int_{R^{|\alpha|+|\beta|}} \Delta_{y-x}^{\beta} D_{w-z}^{\alpha} a(x,z)$$

$$\cdot \prod_{j=1}^{|\alpha|} \psi_{-1/4-\varepsilon}(z_{i_j} - w_{i_j})\, dw_{i_j} \prod_{k=1}^{|\beta|} \psi_{-3/4-\varepsilon}(x_{i_k} - y_{i_k})\, dy_{i_k}$$

for any ε with $0 < \varepsilon < \delta/2$, where $i_1, \ldots, i_{|\alpha|}$ correspond to the nonzero components of α and $i_1, \ldots, i_{|\beta|}$ correspond to the nonzero components of β. The function $\psi_s(x)$, as defined by Cordes in **[3]**, is

$$\psi_s(x) = (2\pi)^{-1/2}\big(2^{1-s}/\Gamma(s)\big)|x|^{s-1/2} K_{s-1/2}(|x|)$$

where Γ and K_σ are the Gamma function and modified Hankel function of order σ. It is the "fundamental solution" of the operator $(1 - d^2/dx^2)^s$.

We use the inequality (7). For $\beta = 0$, we have

$$\left|D_k^{\alpha} a_j'(x,z)\right| \leqslant (\text{constant})|k_1|^{\alpha_1(1/2+\delta)} \cdots |k_n|^{\alpha_n(1/2+\delta)}.$$

By what was proved in **[5]**,

$$A_z(x) \equiv \left(1 - \frac{\partial^2}{\partial z_1^2}\right)^{1/4+\varepsilon} \cdots \left(1 - \frac{\partial^2}{\partial z_n^2}\right)^{1/4+\varepsilon} a_j'(x,z)$$

$$= a_j'(x,z) + \sum_{|\alpha|=1}^{n} \int_{R^{|\alpha|}} D_{w-z}^{\alpha} a_j'(x,z) \prod_{j=1}^{|\alpha|} \psi_{-1/4-\varepsilon}(z_{i_j} - w_{i_j}) dw_{i_j}.$$

Let z be fixed. Then, applying inequality (7) to the above expression we get

$$\left|\Delta_h^{\beta} A_z(x)\right| \leqslant (\text{constant})|h_1|^{\beta_1(3/2+\delta)} \cdots |h_n|^{\beta_n(3/2+\delta)}.$$

By a result similar to what was proved in **[5]**,

$$b_j'(x,z) = \left(1 - \frac{\partial^2}{\partial x_1^2}\right)^{3/4+\varepsilon} \cdots \left(1 - \frac{\partial^2}{\partial x_n^2}\right)^{3/4+\varepsilon} A_z(x)$$

$$= A_z(x) + \sum_{|\beta|=1}^{n} \int_{R^{|\beta|}} \Delta_{y-x}^{\beta} A_z(x) \prod_{k=1}^{|\beta|} \psi_{-3/4-\varepsilon}(x_{i_k} - y_{i_k}) dy_{i_k}.$$

Plugging in $A_z(x)$ gives the desired representation for b_j'. The Lemma follows readily.

The remainder of the proof of the Theorem is an *exact* copy of Kato's proof, part II, given in **[2]**. The only change is that we use

$$g(x,z) = \psi_{3/4+\varepsilon}(x_1) \cdots \psi_{3/4+\varepsilon}(x_n) \psi_{1/4+\varepsilon}(z_1) \cdots \psi_{1/4+\varepsilon}(z_n)$$

instead of $g(x, z) = \psi_{n,t}(x)\psi_{n,s}(x)$ where $s > n/2$, $t > n/2 + 1$. ($\psi_{n,s}$ is the "fundamental solution" of $(1 - \Delta)^{s/2}$ where Δ is the n-dimensional Laplacian.) We have $3/4 + \varepsilon$ for the x variables so that not only $g(X, D)$, the operator with symbol g, will have an extension G in trace class on $L^2(R^n)$, but also $|D|G$ will be in trace class. (See [2].) Q.E.D.

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