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STRATIFIABLE σ-DISCRETE SPACES ARE M_1

GARY GRUENHAGE

ABSTRACT. It is shown that stratifiable σ-discrete spaces are M_1. A corollary is that scattered stratifiable spaces are M_1.

1. Introduction. In 1961, J. Ceder [1] defined the M_i-spaces, $i = 1, 2, 3$. Recently, H. Junnila [4] and the author [2] independently proved that the M_3-spaces (i.e., stratifiable spaces) and the M_2-spaces are actually the same class of spaces. The main question, of course, is whether M_3-spaces are M_1, that is, whether every stratifiable space has a σ-closure-preserving base. The only partial result in this direction is the author's proof in [2] that countable stratifiable spaces are M_1. Here we generalize this result to show that σ-discrete stratifiable spaces are M_1. When combined with a result of P. Nyikos on scattered spaces, this proves also that scattered stratifiable spaces are M_1. But perhaps the main value of this result is the simplicity of the proof relative to the published proof for the countable case, which uses all the machinery of the author's proof that $M_3 \to M_2$.

2. Definitions and main results. A collection $\mathcal{H}$ of subsets of a space X is *closure-preserving* if whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathrm{Cl}(\bigcup \mathcal{H}') = \bigcup \{\overline{H} | H \in \mathcal{H}'\}$. A space X is an M_1-space if it has a σ-closure-preserving base of open sets.

X is an *M_3-space* (or *stratifiable space*) if, to each open $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

(a) $\overline{U}_n \subset U$,

(b) $\bigcup_{n=1}^{\infty} U_n = U$,

(c) $U_n \subset V_n$ whenever $U \subset V$.

A space X is *monotonically normal* [3] if for every pair of disjoint closed sets H and K, there exists an open set $D(H, K)$ containing H such that $\overline{D(H, K)} \cap K = \varnothing$, and if $H' \supset H$ and $K' \subset K$, then $D(H', K') \supset D(H, K)$.

The only fact about stratifiable spaces used in the proof of our main result is that they are monotonically normal and hence collectionwise normal [3].

THEOREM 1. *A σ-discrete stratifiable space is an M_1-space.*

PROOF. Let $X = \bigcup_{n=1}^{\infty} F_n$, where X is stratifiable and each F_n is a closed

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discrete subset of X. Since X is collectionwise normal, it is easy to see that we need only prove that each point of X has a closure-preserving base.

To this end, let $p \in X$. Let D be a monotone normality operator for X. Let us assume that if $n \neq m$, then $F_n \cap F_m = \varnothing$. For each $x \in X$, let $n(x)$ be such that $x \in F_{n(x)}$. We shall define, for each $x \neq p$, a set U_x containing x such that

(i) $\{U_x: x \in F_n\}$ is a discrete collection of clopen sets;

(ii) $U_x \subset D(\{x\}, \{p\} \cup (\bigcup_{i<n(x)} F_i))$;

(iii) if $y \in U_x$, then $U_y \subset U_x$.

To begin with, we use the collectionwise normality of X, and the fact that normal σ-discrete spaces are zero-dimensional, to obtain a collection $\{U_x: x \in F_1\}$ of clopen sets satisfying (i) and (ii). Suppose U_x has been defined for all $x \in \bigcup\{F_i: i < n\}$. Then let $\{U_x: x \in F_n\}$ be a collection satisfying (i) and (ii), and such that $U_x \subset \bigcap\{U_y: n(y) < n \text{ and } x \in U_y\}$.

If H is a closed set with $p \notin H$, let $U(H) = \bigcup\{U_x: x \in H\}$. We claim that $\mathcal{H} = \{X \setminus U(H): H \text{ closed}, p \notin H\}$ is a closure-preserving base at p.

Since $p \in X \setminus U(H) \subset X \setminus H$ whenever H is a closed set not containing p, it will be enough to show that $\mathcal{H}$ is a closure-preserving collection of open sets. To that end, suppose $x \notin \bigcup\{X \setminus U(H_\alpha): \alpha \in A\}$, where each H_α is a closed set not containing p. Then $x \neq p$, and for each $\alpha \in A$, $x \in U_{x(\alpha)}$ for some $x(\alpha) \in H_\alpha$. Then $U_x \subset U_{x(\alpha)}$ for each α, so that $U_x \cap (\bigcup\{X \setminus U(H_\alpha): \alpha \in A\}) = \varnothing$. Thus $\mathcal{H}$ is closure-preserving. We shall complete the proof by showing that $U(H)$ is a clopen set. Suppose $y \notin U(H)$. Then $D(H, \{y\}) \supset D(\{x\}, \{p\} \cup (\bigcup_{i<n(x)} F_i))$ for all $x \in H$ with $n(x) > n(y)$, and in case $y = p$, it is true for all x. Thus $D(H, \{y\}) \supset \bigcup\{U_x: x \in H, n(x) > n(y)\}$, and so $y \notin \mathrm{Cl}(\bigcup\{U_x: x \in H, n(x) > n(y)\})$. But $\bigcup\{U_x: x \in H, n(x) \leqslant n(y)\}$ is a clopen set, so $y \notin \overline{U(H)}$. Thus $U(H)$ is clopen and this completes the proof.

Nyikos [5, Theorem 3.9] has shown that a scattered semistratifiable space is σ-discrete. Our next theorem is a direct consequence of this and Theorem 1.

THEOREM 2. *Every scattered stratifiable space is M_1.*

Finally, we pose a question which, if there is an affirmative answer, would significantly generalize Theorem 1 and would perhaps be easier to solve than the more general question of whether $M_3 \rightarrow M_1$.

QUESTION. Is a stratifiable space having a σ-discrete network consisting of compact sets an M_1-space?

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36830