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TRIANGULATION OF STRATIFIED OBJECTS[1]

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ABSTRACT. A procedure for triangulating an abstract prestratified set is outlined.

1. Introduction. The purpose of this article is to give a short accessible outline of how to construct, for any abstract prestratified set W (see Mather, [7]), a simplicial complex $K \subset \mathbf{R}^N$ and a homeomorphism $f\colon |K| \to W$ which is locally a smooth triangulation of each stratum. In particular, if X is a subanalytic set, it admits the structure of a Whitney stratified object, and therefore of an abstract prestratified set. Therefore X can be triangulated (Hardt, [2]).

Triangulation theorems for stratified objects have been obtained independently by Hendricks [3] (unpublished), Johnson [5] (unpublished), and Kato [6] (in Japanese). I am grateful to R. McPherson for helping to clarify the ideas in the proof. I also wish to thank the Editor, R. Schultz, and an anonymous referee for several valuable suggestions and for pointing out an error in the original version of this paper.

2. Families of lines. Let W be an abstract (or *Thom-Mather*) stratified object (Thom [10], Mather [7]). For each stratum X, let T_X denote the "tubular neighborhood" of X in W, $\pi_X\colon T_X \to X$ the retraction map, and $\rho_X\colon T_X \to [0, \infty)$ the "tubular distance function." Let

$$\begin{aligned} S_X(\varepsilon) &\equiv \{p \in T_X \mid \rho_X(p) = \varepsilon\},\\ T_X(\varepsilon) &\equiv \{p \in T_X \mid \rho_X(p) < \varepsilon\},\\ T_i(\varepsilon) &\equiv \cup\{T_X(\varepsilon) \mid \dim(X) \leqslant i\},\\ W^{(p)} &= \cup\{X \mid \dim(X) \leqslant p\},\\ X_\varepsilon^0 &= X - T_{i-1}(\varepsilon) \quad \text{where } \dim(X) = i,\\ S_\varepsilon^p(X) &= X_\varepsilon^0 \cap \left(\cup\{S_Y(\varepsilon) \mid \dim(Y) \leqslant p\}\right). \end{aligned}$$

Adjusting ρ_X by a positive scale factor $f\colon X \to \mathbf{R}$ by setting $\rho'_X(x) = f(\pi_X(x))\rho_X(x)$ we may assume $\overline{S_X(\varepsilon)}$ is a Thom-Mather stratified object (with tubular projections $\pi_{Y \cap \overline{S_X(\varepsilon)}} = \pi_Y | \overline{Y \cap S_X(\varepsilon)}$ etc.). Note that π_X extends continuously to the closure $\overline{S_X(\varepsilon)} \to \overline{X}$.

Define a *family of lines* on W to be a number $\delta > 0$ together with a system

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of radial projections for each stratum X,

$$r_X(\varepsilon)\colon T_X - X \to S_X(\varepsilon)$$

defined whenever $0 < \varepsilon < \delta$, which satisfy the following commutation relations: If $X < Y$ (i.e., if $X \subset \overline{Y}$) then

(1) $r_X(\varepsilon) \circ r_Y(\varepsilon') = r_Y(\varepsilon') \circ r_X(\varepsilon) \in S_X(\varepsilon) \cap S_Y(\varepsilon')$ for all ε',

(2) $\rho_X \circ r_Y(\varepsilon) = \rho_X$,

(3) $\rho_Y \circ r_X(\varepsilon) = \rho_Y$,

(4) $\pi_X \circ r_Y(\varepsilon) = \pi_X$,

(5) if $0 < \varepsilon < \varepsilon' < \delta$ then $r_X(\varepsilon') \circ r_X(\varepsilon) = r_X(\varepsilon')$,

(6) $\pi_X \circ r_X(\varepsilon) = \pi_X$,

(7) $r_X(\varepsilon)|T_X(\varepsilon) \cap Y\colon T_X(\varepsilon) \cap Y \to S_X(\varepsilon) \cap Y$ is smooth.

For each stratum X we define a stratum preserving homeomorphism

$$h_X\colon T_X - X \to S_X(\varepsilon) \times (0, \infty)$$

by

$$h_X(p) = (r_X(\varepsilon)(p), \rho_X(p)).$$

h_X identifies $T_X(\varepsilon)$ with the mapping cylinder of $S_X(\varepsilon) \to X$.

PROPOSITION 1. *Every Thom-Mather stratified object W admits a family of lines.*

PROOF. $r_X(\varepsilon)$ is constructed by increasing induction on the dimension of the stratum X, with the case $\dim(X) = -1$ trivial. Suppose $r_Y(\varepsilon)$ has been defined for each stratum Y of dimension $\leqslant i - 1$ and suppose X is a stratum of dimension i. Mather [5] constructs for $\varepsilon > 0$ sufficiently small, a projection $T_X - X \to S_X(\varepsilon)$ which is a candidate for $r_X(\varepsilon)$ but which must be altered so as to satisfy condition (1) with respect to all strata $Y < X$. Suppose inductively that $r_X(\varepsilon)$ has been defined and satisfies condition (1) with respect to each stratum Z where $q < \dim(Z) \leqslant i - 1$ and let Y be a stratum of dimension q. Fix $\varepsilon' > 0$. Redefine r_X in the region $T_Y(\varepsilon') \cap T_X$ by setting

$$r'_X(\varepsilon)(p) = h_Y^{-1}\left(r_Y\left(\frac{\varepsilon'}{2}\right)r_X(\varepsilon)h_Y^{-1}\left(r_Y\left(\frac{\varepsilon'}{2}\right)(p), \phi(\rho_Y(p))\right), \rho_Y(p)\right)$$

where

$$h_Y\colon T_Y - Y \to S_Y(\varepsilon'/2) \times (0, \infty),$$
$$h_Y(p) = (r_Y(\varepsilon'/2)(p), \rho_Y(p))$$

and where $\phi\colon \mathbf{R} \to \mathbf{R}$ is a smooth nondecreasing function with

$$\phi(x) = \varepsilon'/2 \quad \text{if } x \leqslant \varepsilon'/2$$

and

$$\phi(x) = x \quad \text{if } x > \tfrac{2}{3}\,\varepsilon'.$$

It is now clear that $r'_X(\varepsilon)$ satisfies the necessary commutation relations with

π_Y, ρ_Y, and r_Y in the region $T_X(\varepsilon) \cap T_Y(\varepsilon'/2)$ and that it continues to satisfy conditions (2)–(5) with respect to all other strata. Furthermore, if $Y < Z < X$ then $r'_X r_Z = r_Z r'_X$ concluding the nested induction. Q.E.D.

(This procedure is essentially described in Hendricks [3].)

3. Triangulations. By "polyhedron" we mean "Euclidean polyhedron" as in Hudson [4]. However if K and L are polyhedra and $J \subset K$ is a subpolyhedron and if f: $J \to L$ is a P.L. embedding then we can define a new polyhedron $K \cup_J L$ by finding triangulations A, B, C of J, K and L such that A is a full subcomplex of B and C and then re-embedding the abstract simplicial complex $B \cup_A C$ into Euclidean space.

DEFINITION. A smooth triangulation of a manifold X is a polyhedral pair (K, L) and a homeomorphism f: $K - L \to X$ such that there exists a simplicial pair (A, B) with $|A| = K$, $|B| = L$ such that for each simplex $\sigma \in A$ and for each point $p \in \bar{\sigma} - |B|$ there is a neighborhood U of p (in the plane containing σ) and a smooth embedding $\tilde{f}$: $U \to X$ which extends $f|(U \cap \bar{\sigma})$.

DEFINITION. A triangulation of a stratified object W is a polyhedron K and a homeomorphism f: $K \to W$ such that, for each stratum X, $f^{-1}(\bar{X})$ is a subpolyhedron of K, and $f^{-1}(X) \to X$ is a smooth triangulation.

For the remainder of this paper we will assume W is a Thom-Mather stratified object with a family of lines and $d > 0$ is sufficiently small that, for each stratum Y, the intersection of any collection of link bundles $Y \cap S_{X_1}(d) \cap \cdots \cap S_{X_k}(d)$ is transverse to the intersection of any disjoint collection $Y \cap S_{Z_1}(d) \cap \cdots \cap S_{Z_l}(d)$. Thus, Y_d^0 is a manifold with corners.

REMARK. In order to triangulate a stratified object W, we will first triangulate the "interior" X_d^0 of each stratum X and then glue on the mapping cylinders of the projections $S_Y(d) \to Y$.

DEFINITION. An interior d-triangulation of an n-dimensional stratified object W^n is a collection of disjoint polyhedra $\{K_X\}$ indexed by the strata X of W, together with an embedding

$$f: K \cup \bigcup_X K_X \to W$$

such that, for each stratum X (say $\dim(X) = i$),

(1) $f(K_X) = X_d^0 \subset X$,

(2) $f|K_X$ is smooth,

(3) $f^{-1}(S_X(d)) = f^{-1}(S_X(d) - T_{i-1}(d))$ is a subpolyhedron of K,

(4) $f^{-1} \circ \pi_X \circ f$: $f^{-1}(S_X(d)) \to f^{-1}(X)$ is P.L.

(It follows that $f^{-1}(\overline{S_X(d)}) \to \overline{S_X(d)}$ is an interior d-triangulation of $\overline{S_X(d)}$.)

A partial (d, p)-triangulation of W^n is an embedding f: $K \to W$ satisfying (2), (3), and (4) as above, but with (1) replaced by

$$f(K_X) = \begin{cases} X_d^0 & \text{if } \dim(X) \leqslant n - 1, \\ S_d^p(X) & \text{if } \dim(X) = n. \end{cases} \tag{1'}$$

If $f\colon K \to W^n$ is a partial (d, p)-triangulation and if $d' < d$ we define the *d'-extension* of (K, f) with respect to X to be the following embedding $g\colon L \to W$:

Let $J = f^{-1}(S_X(d))$ and define $f_1\colon J \times [d', d] \to \overline{T_X(d)} - T_X(d')$ by $f_1(p, t) = r_X(t)(f(p))$.

Let $L = K \cup_{J \times \{d\}} J \times [d', d]$ and set $g|K = f$ and $g|(L - K) = f_1$.

Taking d'-extensions with respect to all strata X (where $\dim(X) \leqslant n - 1$) results in an embedding $h\colon H \to W$ (independent of the order in which the X were chosen because of the commutation relations). Then if Z is the n-dimensional stratum of W, $h^{-1}(W^{(n-1)} \cup S^p_{d'}(Z)) \to W$ is called the partial (d', p)-triangulation induced by (K, f).

Theorem. *Every stratified object has an interior d-triangulation for d sufficiently small.*

Proof. Let B_n and $C_{n,p}$ be the propositions stated below. We shall prove these propositions in the following order:

$$B_{n-1};\ C_{n,0};\ C_{n,1};\ \cdots\ ;C_{n,n-1};\ B_n;\ C_{n+1,0};\ \cdots .$$

Proposition B_n. *Let W^n be an n-dimensional stratified object. Suppose f: $K \to W^{(n-1)}$ is an interior d-triangulation. Let $d' < d$. Then there is an interior d'-triangulation $g\colon L \to W$ such that K is a subpolyhedron of L and $g|K = f$.*

Proposition $C_{n,p}$. *Let W^n be an n-dimensional stratified object and suppose $f\colon K \to W$ is a partial $(d, p - 1)$-triangulation. Let $d' < d$. Let $j\colon J \to W$ be the $(d', p - 1)$-partial triangulation induced from (K, f). Then there is a partial (d', p)-triangulation $h\colon H \to W$ so that J is a subpolyhedron of H and $j|J = j$.*

Proof of Proposition B_n. Given an interior d-triangulation $f\colon K \to W^{(n-1)}$, let $d' < d'' < d$ and apply Proposition $C_{n,0}$ through $C_{n,n-1}$ to obtain a partial $(d'', n - 1)$-triangulation $g\colon L \to W$ such that $K \subset L$ and $g|L = f$. Let (L', g') be the d'-extension of (L, g) taken with respect to all strata of dimension $\leqslant n - 1$. Suppose Z is the n-dimensional stratum. Then $g'(L') \supset Z^0_{d'} - Z^0_{d''}$. Choose a smooth embedding $\beta\colon N \to Z$ of a simplicial complex N so that $Z^0_{d''} \subset \beta(N) \subset Z^0_{d'}$. Then, as in Munkres [8] there is a refinement N' of N and an approximation β' of β so that (N', β') and (L', g') fit together in a triangulation of $Z^0_{d'}$. Thus, gluing N' and L' along their common intersection gives the desired d'-triangulation.

Proof of Proposition $C_{n,p}$. Given a partial $(d, p - 1)$-triangulation $f\colon K \to W$, let $L_0 = f^{-1}(\overline{S_X(d)})$ where $\dim(X) = p$. Then $(L_0, f|L_0)$ is a partial $(d, p - 2)$-triangulation of $\overline{S_X(d)}$. Let $d' < d'' < d$ and let $\alpha_1\colon K_1 \to W$ be the d''-extension of (K, f) taken with respect to all strata Y where $Y \neq X$. Let Z be the n-dimensional stratum. Then $L_1 = \alpha_1^{-1}(\overline{S_X(d)} \cap (W^{(n-1)} \cup S^{p-1}_{d''}(Z)))$ is a subpolyhedron of K_1 and $\alpha_1^{-1} \circ \pi_X \circ \alpha_1\colon L_1 \to K_1$ is piecewise linear.

Since (L_1, α_1) is the $(d'', p - 2)$-partial triangulation of $\overline{S_X(d)}$ induced by

$(L_0, f|L_0)$ we may apply Proposition $C_{n-1,p-1}$ through $C_{n-1,n-2}$ to find a partial $(d'', n-2)$-triangulation g: $M \to \overline{S_X(d)}$ such that L_1 is a subpolyhedron of M, and $g|L_1 = \alpha_1$.

Note that $\alpha_1^{-1} \circ \pi_X \circ g$: $M \to \alpha_1^{-1}(X)$ is P.L., for if $Y > X$ then π_X can be locally factored:

$$\begin{tikzcd}
S_X(d) \cap S_Y(d'') \arrow[r, "\pi_Y"] & S_X(d) \cap Y \arrow[r, "\pi_X"] & X \\
g^{-1}(S_X(d) \cap S_Y(d'')) \arrow[u, "g"'] \arrow[r, "\tilde{\pi}_Y"] & \alpha_1^{-1}(S_X(d) \cap Y) \arrow[u, "\alpha_1"'] \arrow[r, "\tilde{\pi}_X"] & \alpha_1^{-1}(X) \arrow[u, "\alpha_1"']
\end{tikzcd}$$

where $\tilde{\pi}_Y$ and $\tilde{\pi}_X$ are P.L.

We now triangulate the rest of $Z \cap S_X(d)$ so as to make π_X piecewise linear, where Z is the n-dimensional stratum in W.

Let (M_2, g_2) be the d'-extension of (M, g) taken with respect to all strata in $\overline{S_X(d)}$. Thus $g_2(M_2) \supset (Z^0_{d'} - Z^0_{d''}) \cap S_X(d)$. By Putz **[9]** there is a polyhedron N and a smooth embedding β: $N \to Z \cap S_X(d)$ so that

$$\beta(N) \supset Z^0_{d''} \text{ and } \alpha_1^{-1} \circ \pi_X \circ \beta\colon N \to \alpha_1^{-1}(X) \text{ is P.L.} \tag{$*$}$$

According to Putz, (N, β) can be approximated by an embedding (N_2, β_2) so that $(*)$ continues to hold and so that (M_2, g_2) and (N_2, β_2) intersect in a subpolyhedron P, i.e.,

$$P = g_2^{-1}(\beta_2(N_2)) = \beta_2^{-1}(g_2(M_2)).$$

Take $L = N_2 \cup_P M_2$ and let γ: $L \to \overline{S_X(d')}$ by

$$\gamma|M_2 = r_X(d') \circ g_2; \quad \gamma|N_2 = r_X(d') \circ \beta_2.$$

Finally, let (K_2, α_2) be the partial $(d', p-1)$-triangulation of W induced by (K, f). The commutation relations guarantee that (K_2, α_2) and (L, γ) intersect in a subpolyhedron Q, i.e.,

$$Q = \gamma^{-1}\left(\overline{S_X(d')} \cap \left(W^{(n-1)} \cup S_{d'}^{p-1}(Z)\right)\right).$$

Thus, $K_2 \cup_Q L \to W$ is the desired partial (d', p)-triangulation.

4. Triangulation of mapping cylinders. Let f: $K \to L$ be a simplicial map between simplicial complexes. Let L' be a barycentric subdivision of L. Choose a barycentric subdivision K' of K so that $K' \to L'$ is simplicial. (The barycenter of a simplex σ is denoted $\hat{\sigma}$.)

Let S_K be the simplicial mapping cylinder of f in the sense of Cohen **[1]** and let S'_K be the subdivided mapping cylinder, i.e.,

$$S_K = L \cup \{A * \hat{\sigma}_0\hat{\sigma}_1 \cdots \hat{\sigma}_p | A \in L, \sigma_0 < \sigma_1 < \cdots < \sigma_p \in K, \text{ and } A < f(\sigma_0)\},$$

$$S'_K = L' \cup \{\hat{\tau}_0\hat{\tau}_1 \cdots \hat{\tau}_q\hat{\sigma}_0 \cdots \hat{\sigma}_p | \tau_0 < \tau_1 < \cdots < \tau_q \in K, \tau_q < f(\sigma_0), \sigma_0 < \sigma_1 < \cdots < \sigma_p \in L\}.$$

It is good to think of S_K as consisting of the pieces $\{A * f^{-1}(D(A)) | A \in L\}$ where $D(A)$ denotes the dual of A in L.

The simplicial retraction $\tilde{\pi}$: $S'_k \to L'$ is given by $\tilde{\pi}(\hat{\tau}_0 \cdots \hat{\tau}_q\hat{\sigma}_0 \cdots \hat{\sigma}_p) = \hat{\tau}_0 \cdots \hat{\tau}_q f(\hat{\sigma}_0) \cdots f(\hat{\sigma}_p)$. Thus the corresponding retraction $|S_K| \to |L|$ is piecewise linear.

Let M_K be the topological mapping cylinder of F, $M_K = |K| \times [0, 1] \cup |L|/(x, 0) \sim f(x)$ and let π: $M_K \to |L|$ be the retraction $\pi(x, t) = f(x)$.

We will define a continuous H: $|S_K| \to M_K$ such that for each simplex $\sigma \in K$, H takes $|S_\sigma|$ homeomorphically to M_σ and such that $H| \, |S_K - L|$ is smooth. For fixed $\sigma \in K$ suppose the vertices of $\tau = f(\sigma)$ are denoted $v_0, v_1, \ldots, v_n$. For any $y \in \sigma$ there is a unique decomposition $y = \sum_{i=0}^n a_i y'_i$ such that $f(y'_i) = v_i$ for $0 \leqslant i \leqslant n$. This determines $n + 1$ smooth projection maps P_i: $\sigma - f^{-1}(v_0 \ldots \hat{v}_i \ldots v_n) \to f^{-1}(v_i)$ by $P_i(y) = y'_i$.

Now let $A < \tau$ (say, $A = v_0 v_1 \ldots v_k$) and let $B = \hat{\sigma}_0\hat{\sigma}_1 \ldots \hat{\sigma}_p$ where $A < f(\sigma_0)$ and $\sigma_0 < \sigma_1 < \cdots < \sigma_p < \sigma$. Each $x \in A * B$ may be written $x = ty + (1 - t)z$ where $t \in [0, 1]$, $y \in B$ and $z = \sum_{i=0}^k b_i v_i \in A$. Define H': $A * B \to \sigma \times [0, 1]$ by

$$H'(x) = \left(ty + (1 - t)\sum_{i=0}^{k} b_i P_i(y), t\right).$$

Then H' is smooth provided $f(y) \notin \bigcup_{i=0}^k \langle v_0 \ldots \hat{v}_i \ldots v_n \rangle$ which is guaranteed by the property $f(y) \in D(A)$. Composing with the projection $|\sigma| \times [0, 1] \to M_\sigma$ we obtain the desired map H. H commutes with the retraction to $|L|$ and $H|(|K| \cup |L|)$ is the identity.

LEMMA. *H is a homeomorphism.*

PROOF. In the case $\sigma = \tau$ and $f =$ identity, H is linear. A typical simplex $\hat{\sigma}_0\hat{\sigma}_1 \cdots \hat{\sigma}_p * \hat{\sigma}_p\hat{\sigma}_{p+1} \cdots \hat{\sigma}_n$ in S'_σ is mapped onto the region in M_σ given by

$$\left\{\left(\sum_{i=0}^{n} a_i\hat{\sigma}_i, t\right) \in \hat{\sigma}_0 \cdots \hat{\sigma}_n \times [0, 1] \,\middle|\, a_0 + a_1 + \cdots + a_{p-1} \leqslant 1 - t \text{ and } a_0 + a_1 + \cdots + a_p \geqslant 1 - t\right\},$$

Using this fact, H^{-1} is easily found.

In general, if f: $\sigma \to \tau$ is simplicial, let S_τ be the simplicial mapping cylinder of the identity map $\tau \to \tau$ and let M_τ be the topological mapping cylinder. Let H_τ: $|S_\tau| \to M_\tau$ be the above homeomorphism.

We obtain a commuting diagram

$$\begin{array}{ccc} |S_\sigma| & \xrightarrow{H} & M_\sigma \\ F\big\downarrow & & \big\downarrow G \\ |S_\tau| & \xrightarrow{H_\tau} & M_\tau \end{array}$$

where $F(A * \hat{\sigma}_0 \cdots \hat{\sigma}_p) = A * f(\hat{\sigma}_0 \cdots \hat{\sigma}_p)$ and $G(x, t) = (f(x), t)$.

Suppose the vertices of τ are $v_0, v_1, \ldots, v_n$.

For each $y \in f^{-1}(\hat{\tau})$ define an embedding g_y: $\tau \to \sigma$ by $g_y(\Sigma_{i=0}^n b_i v_i) = \Sigma_{i=0}^n b_i P_i(y)$.

This induces sections $\tilde{F}_y$ of F and $\tilde{G}_y$ of G by

$$\tilde{F}_y(A * \hat{\tau}_0 \cdots \hat{\tau}_q) = A * g_y(\hat{\tau}_0 \cdots \hat{\tau}_q)$$

whenever $A \leqslant \tau_0 \ldots . < \tau_q < \tau$, and $\tilde{G}_y(x, t) = (g_y(x), t)$ whenever $x \in \tau$.

These sections, for different $y \in f^{-1}(\hat{\tau})$ decompose the interiors of $|S_\sigma|$ and of M_σ. Furthermore, $H \circ \tilde{F}_y = \tilde{G}_y \circ H_\tau$ so H takes each section in $|S_\sigma|$ homeomorphically to the corresponding section in M_σ. Thus H is a homeomorphism.

5. PROPOSITION. *Every stratified object W^n can be triangulated.*

PROOF. Let f: $K \to W$ be an interior d-triangulation. We prove by descending induction on i that there is a polyhedron J_i and an embedding g_i: $J_i \to W$ so that $K = J_n \subset J_{n-1} \subset \cdots \subset J_i$, $g_i|K = \underline{f}$, $g_i|J_p = g_p$, $g_i(J_i) = f(K) \cup (W - T_{i-1}(d))$ and for each stratum X, $g_i^{-1}(X)$ is a subpolyhedron and $\underline{g_i}^{-1}(X) \to X$ is smooth. We also assume that if $\dim(Y) \leqslant i - 1$ then $g_i^{-1}(\overline{S_Y(d)})$ is a subpolyhedron and $g_i^{-1}(S_Y(d)) \to g_i^{-1}(Y)$ is P.L. Then g_0 will be a triangulation. We start with $g_n = f$.

To construct (J_{i-1}, g_{i-1}) from (J_i, g_i) suppose X is the $(i - 1)$-dimensional stratum. Choose simplicial complexes A and B such that $|A| = g_i^{-1}(S_X(d))$, $|B| = g_i^{-1}(X)$ and $A \to B$ is simplicial. We also demand that if $Y < X$ then $g_i^{-1}(S_X(d) \cap Y)$ and $g_i^{-1}(S_X(d) \cap S_Y(d))$ are full subcomplexes. Choose barycenters for B, associated barycenters for A and let S be the simplicial mapping cylinder of $A \to B$. Let M be the topological mapping cylinder of $|A| \to |B|$ and let F: $|S| \to M$ be the homeomorphism from §4. Define G: $M \to W$ by $G(p, t) = r_X(td)(p)$. Let $P = |A| \cup |B|$ and define $J_{i-1} \equiv J_i \cup_P |S|, g_{i-1}| \, |S| \equiv G \circ F$ and $g_{i-1}|J_i \equiv g_i$. One checks that (J_{i-1}, g_{i-1}) satisfy the induction hypotheses. Then g_0: $J_0 \to W$ is the desired triangulation.

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